Characterizing ω-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs

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1. Context

Strategy synthesis for zero-sum turn-based games

Design **optimal** controllers for systems interacting with an **antagonistic** environment.

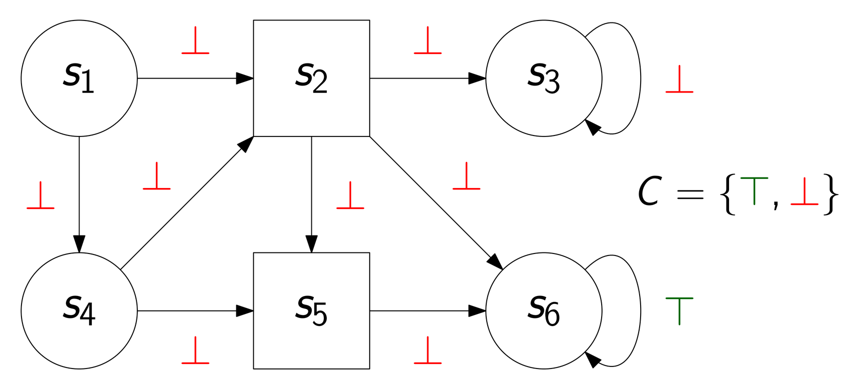
Interest in "simple" controllers

When do **finite-memory** strategies suffice to play optimally? Focus on **infinite** graphs.

Inspiration

Memoryless determinacy in infinite graphs. [CN06]

2. Zero-sum turn-based games on graphs



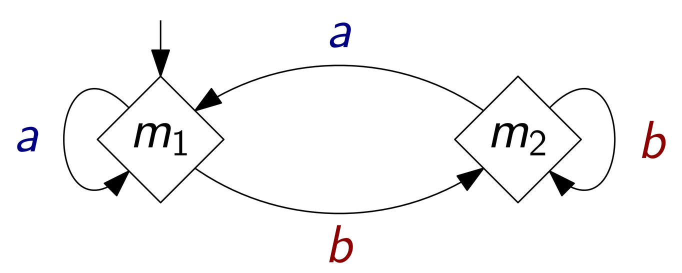
- Two-player **arenas**: S_1 (◯, for $\mathcal{P}_1$) and S_2 (□, for $\mathcal{P}_2$), edges E.
- Set C of **colors**. Edges are colored.
- **Objectives** are sets $W \subseteq C^\omega$. **Zero-sum**.
- A **strategy** for $\mathcal{P}_i$ is a function $\sigma\colon E^* \to E$.

3. Finite memory

Finite-memory strategy $\approx$ memory structure + next-act. function

Definition. *Memory structure* $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$: *finite set of states M, initial state m_{init}, update function $\alpha_{\mathsf{upd}}\colon M \times C \to M$.*

Ex.: remember whether a or b was last played:



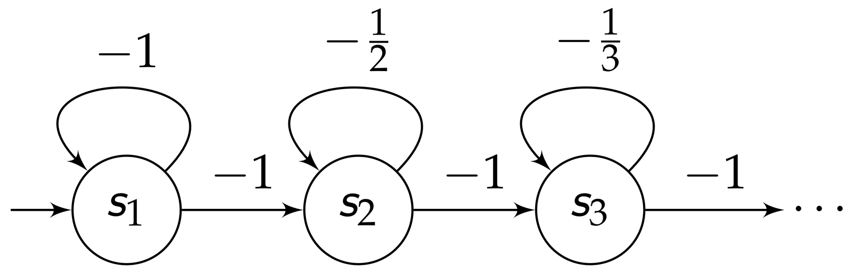
Next-action function $\alpha_{\mathsf{nxt}}\colon S_i \times M \to E$.

Memoryless strategies use **memory structure** $\to\Diamond\circlearrowleft C$.

4. More complex strategies in infinite arenas

Colors $C = \mathbb{Q}$, objective W = "obtain a *mean payoff* ≥ 0".

- **Memoryless** strategies suffice in **finite** arenas. [EM79]
- **Infinite** memory required in **infinite** arenas. [Put94]



$\rightsquigarrow$ Possible to get 0 at the limit **with infinite memory**: loop increasingly many times in states s_n.

5. Myhill-Nerode in languages of infinite words?

Let W be a language of **infinite** words ($=$ an objective) on C.

Right congruence

For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \Leftrightarrow yz \in W$.

Links with ω-regularity?

- If W is **ω-regular**, then $\sim_W$ has finite index. DFA $\mathcal{M}_\sim$ *prefix-classifier* associated with $\sim_W$.
- Reciprocal not true.

6. Four examples

Objective	Prefix-classifier $\mathcal{M}_\sim$	Memory
$C = \{0, \dots, n\}$, Parity condition	$\to\Diamond\circlearrowleft C$	$\to\Diamond\circlearrowleft C \mapsto \{0, \dots, n\}$
$C = \mathbb{Q}$, $W = \mathsf{MP}^{\geq 0}$	$\to\Diamond\circlearrowleft C$	No finite structure
$C = \{a, b\}$, $W = b^*ab^*aC^\omega$	$b,1$; $b,1$; $a,1$; $a,1$; $C,2$	$\to\Diamond\circlearrowleft C$
$C = \{a, b\}$, $W = C^*(ab)^\omega$	$\to\Diamond\circlearrowleft C$	$a,0$; $a,1$; $b,1$; $b,0$

7. Main result [BRV22]

Let $W \subseteq C^\omega$ be an objective.

Theorem

If a memory structure $\mathcal{M}$ suffices to play optimally in **(one-player)** infinite arenas for both players, **then**

- $\mathcal{M}_\sim$ is finite, and
- *W* **is recognized by a parity automaton** $(\mathcal{M}_\sim \otimes \mathcal{M}, p)$.

$\rightsquigarrow$ if $\mathcal{M}_\sim \otimes \mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$,
$p\colon M \times C \to \{0, \dots, n\}$.

Generalizes [CN06], where $\mathcal{M}_\sim = \mathcal{M} = \to\Diamond\circlearrowleft C$.

8. Corollaries

Let $W \subseteq C^\omega$ be an objective.

Characterization

W is **finite-memory-determined** if and only if W is **ω-regular**.

One-to-two-player FM lift (inspired by [GZ05])

W is finite-memory-determined in **one-player** infinite arenas
$\Longrightarrow$ W is finite-memory-determined in **two-player** infinite arenas.

Proof. W is finite-memory-determined in **one-player** arenas
$\xRightarrow{\text{[BRV22]}}$ W is recognized by a det. parity automaton (ω-regular)
$\xRightarrow{\text{[Zie98]}}$ this DPA (as a memory) suffices in **two-player** arenas
$\Longrightarrow$ this DPA (as a memory) suffices in **one-player** arenas.

References

[CN06] Thomas Colcombet and Damian Niwiński. On the positional determinacy of edge-labeled games. *Theor. Comput. Sci.*, 352(1-3):190–196, 2006.

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[GZ05] Hugo Gimbert and Wieslaw Zielonka. Games where you can play optimally without any memory. In Martín Abadi and Luca de Alfaro, editors, *CONCUR 2005 - Concurrency Theory, 16th International Conference*, volume 3653 of *LNCS*, pages 428–442. Springer, 2005.

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