Half-Positional Objectives Recognized by Deterministic Büchi Automata

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Outline

Synthesis problem

Synthesizing **controllers** for **reactive systems** with an **objective**.
Systems and their environment modeled with **zero-sum games**.

Strategy complexity

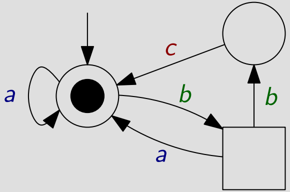
When can we implement **simple and concise** optimal controllers?

Results

Effective characterization of objectives admitting controllers with no need for memory, among those recognizable by **deterministic Büchi automata**.

Games

Zero-sum turn-based games on graphs



- $C = \{a, b, c\}$, $\mathcal{A} = (V_1, V_2, E)$.
- Two players $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$)

Motivation

Understand the **objectives** for which **simple** strategies suffice to win (in all arenas).

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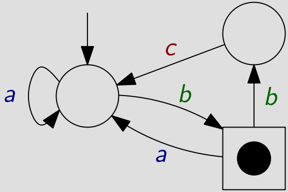

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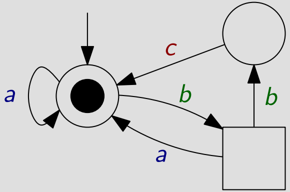

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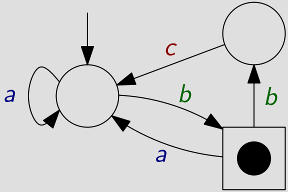

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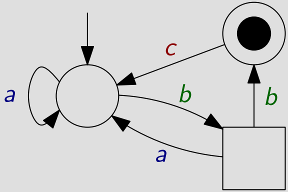

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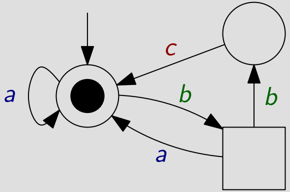

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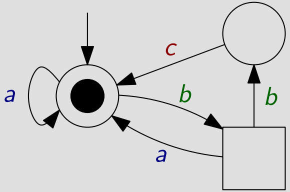

- $C = \{a, b, c\}$, $\mathcal{A} = (V_1, V_2, E)$.
- Two players $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$) generate an infinite word $w = babbc \ldots \in C^\omega$.
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.

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Understand the **objectives** for which **simple** strategies suffice to win (in all arenas).

Half-positionality

Strategies

A **strategy** of $\mathcal{P}_1$ is a function $\sigma \colon E^* \to E$.
It is **positional** if the choices only depend on the **current** vertex, i.e., if

$$\sigma \colon V_1 \to E.$$

Half-positional objectives

In all games with objective W, if $\mathcal{P}_1$ can win with **some** strategy, can $\mathcal{P}_1$ also win with a **positional** strategy?

$\rightsquigarrow$ If yes, W is **half-positional**.

W is **bipositional** if both $\mathcal{P}_1$ (objective W) and $\mathcal{P}_2$ (objective $C^\omega \setminus W$) have positional winning strategies.

Half-positionality

Bipositionality is well-understood

- **Characterization** over finite arenas.[1]
- **Characterization** over infinite arenas.[2]

Previous results on half-positionality

- **Sufficient** conditions for half-positionality over finite arenas.[3,4]
- Structural **characterization** over infinite arenas.[5]

Might still be difficult to **decide** if an objective is half-positional!

[1] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[2] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[3] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006.

[4] Bianco et al., "Exploring the boundary of half-positionality", 2011.

[5] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2022.

Objectives

Common class of objectives finitely representable:
ω-regular objectives.

Open problem

Half-positionality **not** completely understood for ω-regular objectives!

Here

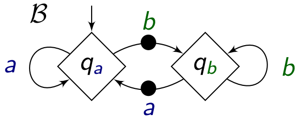
Effective characterization of **half-positional** objectives recognized by **deterministic Büchi automata** (DBA).

DBA recognize a **sub**class of the ω-regular objectives.

Deterministic Büchi automata

A **deterministic Büchi automaton (DBA)** $\mathcal{B}$ on alphabet C

- reads **infinite** words (in C^ω),
- accepts words that see infinitely many **Büchi transitions** •.



$$\mathcal{L}(\mathcal{B}) = \{w \in \{a, b\}^\omega \mid w \text{ sees } \infty\text{ly many } a \text{ and } \infty\text{ly many } b\}$$

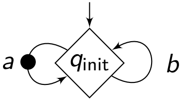
Central question

Given a DBA $\mathcal{B}$, **is objective $W = \mathcal{L}(\mathcal{B})$ half-positional**?

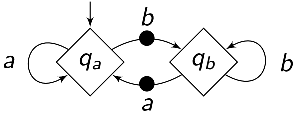
Examples (1/2)

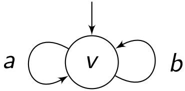
$C = \{a, b\}$.

- $W = \text{Büchi}(a) =$ "seeing a infinitely often": half-positional.[6]



- $W = \text{Büchi}(a) \cap \text{Büchi}(b)$: **not** half-positional.



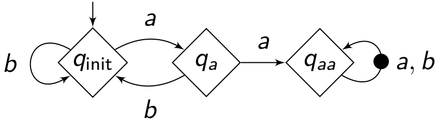


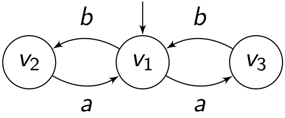
[6] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

Examples (2/2)

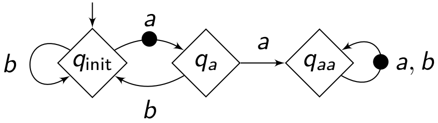
$C = \{a, b\}$.

- $W = C^* aa C^\omega$: **not** half-positional.





- $W = \text{Büchi}(a) \cup C^* aa C^\omega$: **half-positional**.



$\rightsquigarrow$ **Not** bipositional, does **not** satisfy previous sufficient conditions for half-positionality. But encompassed by our characterization!

Characterization of **half-positionality** of
DBA-recognizable objectives
with a conjunction of **three** conditions

Relations on prefixes

Let $W \subseteq C^\omega$ be an objective.

Accepted continuations

For $u \in C^*$, $u^{-1}W = \{w \in C^\omega \mid uw \in W\}$.

For $u, v \in C^*$,

- $u \sim_W v$ if $u^{-1}W = v^{-1}W$ ($\approx$ Myhill-Nerode equivalence relation),
- $u \preceq_W v$ if $u^{-1}W \subseteq v^{-1}W$ (called **prefix preorder**).

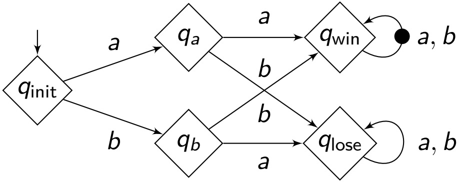
Condition 1: $\preceq_W$ is total

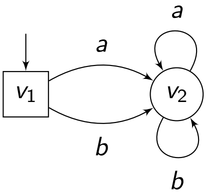
Let $W \subseteq C^\omega$ be an objective.

Condition 1

Prefix preorder $\preceq_W$ is total.

$W = (aa + bb)C^\omega$: words $a, b \in C^*$ are **not** comparable for $\preceq_W$.





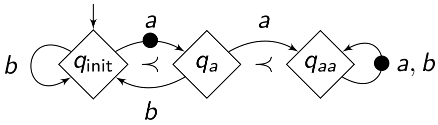
Condition 1: $\preceq_W$ is total

Let $W \subseteq C^\omega$ be an objective.

Condition 1

Prefix preorder $\preceq_W$ is total.

$W = \text{Büchi}(a) \cup C^* aaC^\omega$: total prefix preorder $\preceq_W$.



Condition 2: progress-consistency

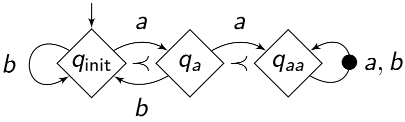
Let $W \subseteq C^\omega$ be an objective.

Condition 2

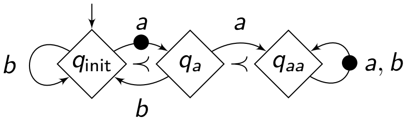
Objective W is **progress-consistent** if

$$\text{for all } u, v \in C^*,\ u \prec_W uv \text{ implies } uv^\omega \in W.$$

C^*aaC^ω **is not** progress-consistent: $b \prec b(ba)$ but $b(ba)^\omega \notin W$.

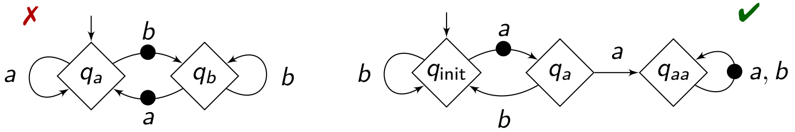


Büchi$(a) \cup C^*aaC^\omega$ **is** progress-consistent (here, $b(ba)^\omega \in W$).



Intuition for third condition

What distinguishes these two DBA?



- Left: needs two states for the objective, but same accepting continuations ($q_a \sim q_b$; same objective when taken as initial states).
- **Right: three states, all recognizing different objectives**.

Being "**Myhill-Nerode**-like" is necessary for half-positionality.

Condition 3: one state per equivalence class

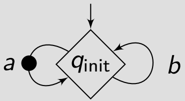
Let $W \subseteq C^\omega$ be an objective recognized by a DBA.

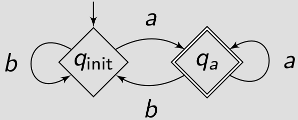
Condition 3

Objective W is **Myhill-Nerode-like** if it can be recognized by a DBA with one state per equivalence class of $\sim_W$.

Remark

We consider **transition**-based acceptance, not **state**-based!
$W = \text{Büchi}(a)$ only satisfies the condition for transition-based:





Characterization

Theorem

An objective W **recognized by a DBA** is **half-positional if and only if**

- $\preceq_W$ is total,
- W is progress-consistent, and
- W is Myhill-Nerode-like.

All three conditions are decidable in **polynomial** time (by checking containment/emptiness of deterministic automata).

Conclusion

Let W be an objective recognized by a DBA $\mathcal{B}$.

Corollary 1: Polynomial-time algorithm

Half-positionality of W can be **decided** in $\mathcal{O}(|\mathcal{B}|^4)$ time.

Corollary 2: Finite-to-infinite, one-to-two-player lift

If W is half-positional over **finite one-player** games, then also over **infinite two-player** games!

Future works

- Half-positionality in all ω-regular objectives?
- More generally, memory requirements?

Thanks!