Half-Positional Objectives Recognized by Deterministic Büchi Automata

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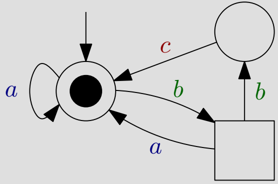
Games on graphs

Zero-sum turn-based games on graphs



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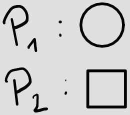

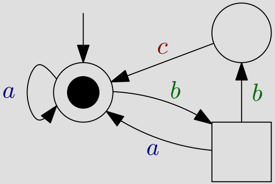

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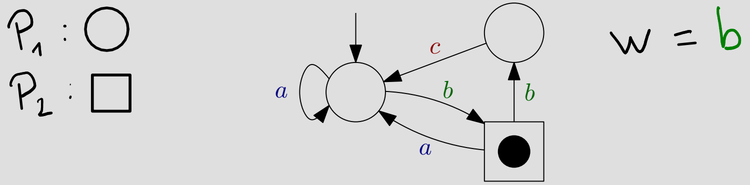

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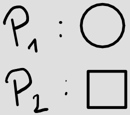

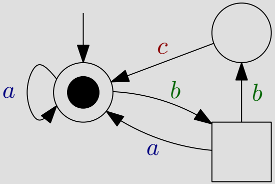

w = ba

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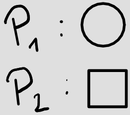

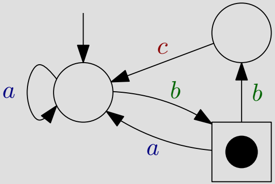

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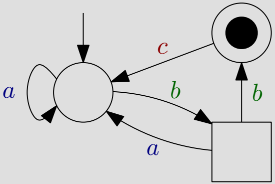
P_1 : ○

P_2 : □



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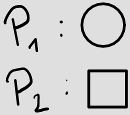

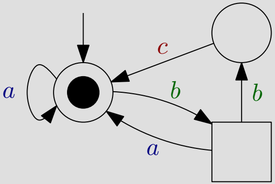

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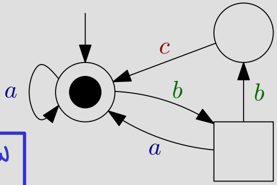
Objectives: $W \subseteq C^\omega$



$w = babbc\ldots$

$C = \{a, b, c\}$

Games on graphs

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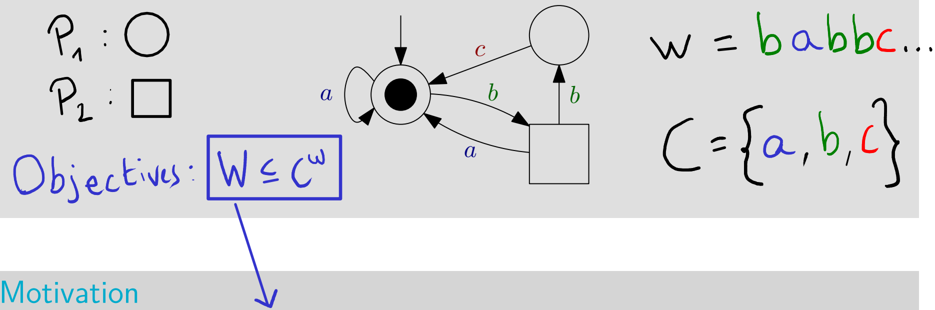

Motivation

Understand the **objectives** for which **simple** strategies suffice to win.

Half-positionality

Strategies

A **strategy** of $\mathcal{P}_1$ is a function $\sigma \colon E^* \to E$.

It is **positional** if the choices only depend on the **current** vertex.

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In all games, if $\mathcal{P}_1$ can win for objective W with a strategy $\sigma\colon E^* \to E$, can $\mathcal{P}_1$ also win with a **positional** strategy?

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Existing results

- **Sufficient** conditions for half-positionality.[1,2]
- Structural **characterization**![3]

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[1] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006.

[2] Bianco et al., "Exploring the boundary of half-positionality", 2011.

[3] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2022.

Objectives

Common class of objectives finitely representable:
ω-**regular objectives**.

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Open problem

Half-positionality **not** completely understood for ω-regular objectives!

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Here

Effective characterization of **half-positional** objectives recognized by **deterministic Büchi automata** (DBA).

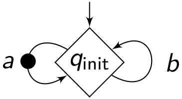
DBA recognize a **sub**class of the ω-regular objectives.

Examples


"Büchi transitions"

$C = \{a, b\}$.

- $W = \text{Büchi}(a) =$ "seeing a infinitely often": half-positional.[4]



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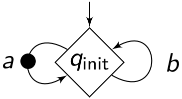
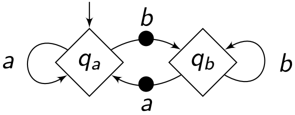
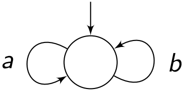


- $W = \text{Büchi}(a) \cap \text{Büchi}(b)$: **not** half-positional.





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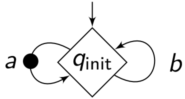

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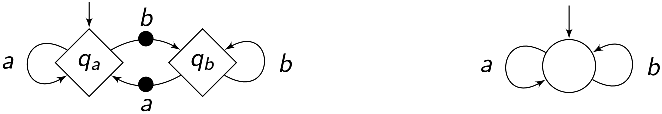
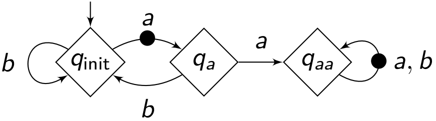


- $W = \text{Büchi}(a) \cup C^* aa C^\omega$: half-positional.



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Characterization

Let W be recognized by a DBA $\mathcal{B}$.

Main result

Characterization of half-positionality of W with a conjunction of **three** conditions.

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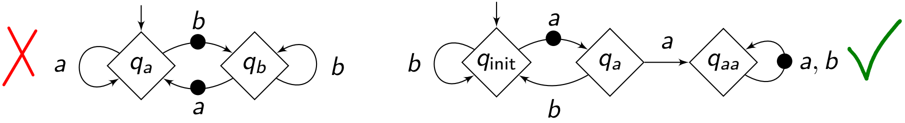
Intuition for **one** condition: what distinguishes these two DBA?



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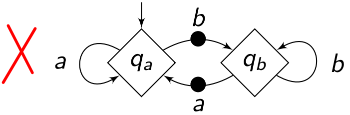

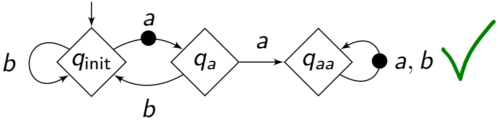

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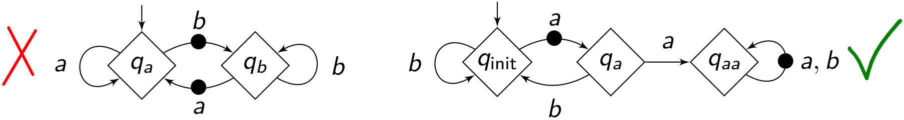

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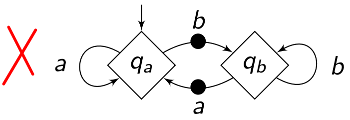

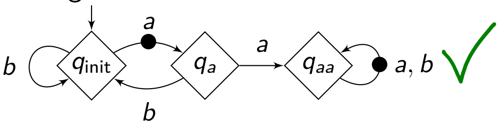

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Being "**Myhill-Nerode**-like" is necessary for half-positionality.

Conclusion: two corollaries

Let W be recognized by a DBA $\mathcal{B}$.

Polynomial-time algorithm

Half-positionality of W can be **decided** in $\mathcal{O}(|\mathcal{B}|^4)$ time.

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If W is half-positional over **finite one-player** graphs, then also in **infinite two-player** games!

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Thanks!

Appendix

Relations on prefixes

Let $W \subseteq C^\omega$ be an objective.

Left quotient

For $u \in C^*$, $u^{-1}W = \{w \in C^\omega \mid uw \in W\}$.

For $u, v \in C^*$,

- $u \sim v$ if $u^{-1}W = v^{-1}W$ ($\approx$ Myhill-Nerode relation),
- $u \preceq v$ if $u^{-1}W \subseteq v^{-1}W$.

Condition 1: $\preceq$ is total

Let $W \subseteq C^\omega$ be an objective.

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Prefix preorder $\preceq$ is total.

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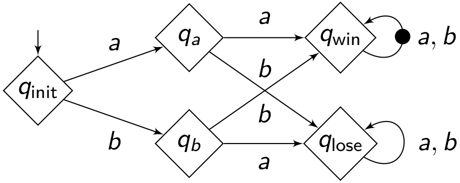
For $W = (aa + bb)C^\omega$, words a and b are not comparable for $\preceq$.



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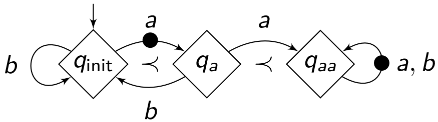
$\text{Büchi}(a) \cup C^* aaC^\omega$ has a total prefix preorder.



Condition 2: progress-consistency

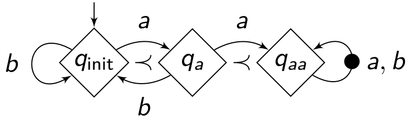
Let $W \subseteq C^\omega$ be an objective.

Condition 2

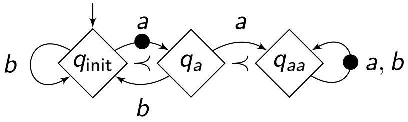
Objective W is **progress-consistent** if

$$\text{for all } u, v \in C^*,\ u \prec uv \text{ implies } uv^\omega \in W.$$

$C^* aa C^\omega$ is **not** progress-consistent: $b \prec b(ba)$ but $b(ba)^\omega \notin W$.



Büchi$(a) \cup C^* aa C^\omega$ is progress-consistent (here, $b(ba)^\omega \in W$).



Condition 3: one state per class

Let $W \subseteq C^\omega$ be an objective recognized by a DBA.

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Objective W is **Myhill-Nerode-like** if it is recognized by a DBA with one state per equivalence class for $\sim$.

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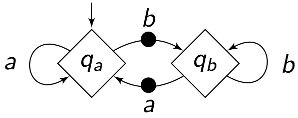
$\text{Büchi}(a) \cap \text{Büchi}(b)$ is **not** Myhill-Nerode-like. **One** equivalence class, but needs at least **two** states.



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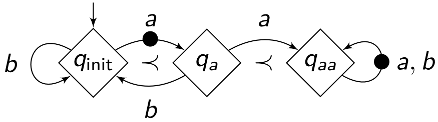
$\text{Büchi}(a) \cup C^* aaC^\omega$ is Myhill-Nerode-like (**three** classes, **three** states).



Characterization

Theorem

An objective W recognized by a DBA is half-positional **if and only if**

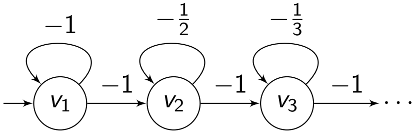
- $\preceq_W$ is total,
- W is progress-consistent, and
- W is Myhill-Nerode-like.

All three conditions are easy to decide.

Greater memory requirements in **infinite** arenas in general

Objective: get a mean payoff ≥ 0.

- Memoryless strategies suffice in finite (even stochastic) arenas.
- Infinite memory is required in one-player deterministic infinite arenas.[5]



$\rightsquigarrow$ Possible to get 0 at the limit **with infinite memory**.

[5] Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, 1994.