Characterizing ω-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs

Pierre Vandenhove[1,2]

Joint work with Patricia Bouyer[2] and Mickael Randour[1]

[1]F.R.S.-FNRS & UMONS – Université de Mons, Belgium

[2]Université Paris-Saclay, CNRS, ENS Paris-Saclay, LMF, France

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Outline

Synthesis problem

Synthesizing **controllers** for **reactive systems** with an **objective**.
Systems and their environment modeled with **zero-sum games**.

Strategy complexity

Finite-memory determinacy: when do **finite-memory** strategies suffice?
Focus on games played on **infinite** graphs.

Inspiration

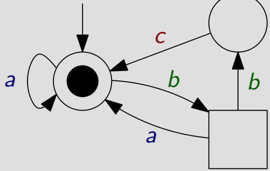
Results about **memoryless determinacy** in finite[1] and infinite[2] graphs.

[1] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[2] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Games

Zero-sum turn-based games on graphs



- $C = \{a, b, c\}$, $\mathcal{A} = (S_1, S_2, E)$.
- Two players $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$)

Motivation

Understand the **objectives** for which **simple** strategies suffice to win (in all arenas).

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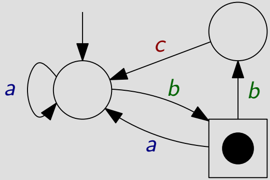

- $C = \{a, b, c\}$, $\mathcal{A} = (S_1, S_2, E)$.
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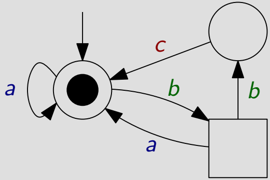

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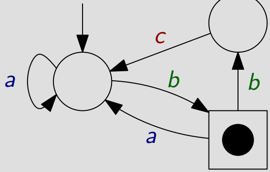

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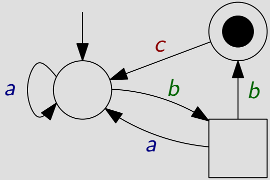

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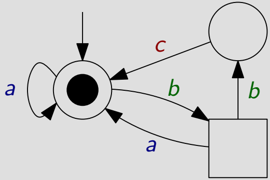

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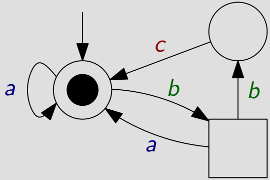

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- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.

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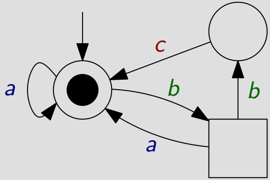
Zero-sum turn-based games on graphs



- $C = \{a, b, c\}$, $\mathcal{A} = (S_1, S_2, E)$.
- Two players $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$) generate an infinite word $w = babbc\ldots \in C^\omega$.
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.
- **Strategy** for $\mathcal{P}_i$: function $\sigma\colon E^* \to E$.

Motivation

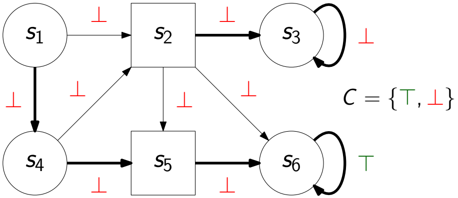
Understand the **objectives** for which **simple** strategies suffice to win (in all arenas).

Memoryless determinacy

Motivation

Understand the **objectives** for which **simple** strategies suffice to win.

A strategy σ of $\mathcal{P}_i$ is *memoryless* if it is a function ~~E^*~~ $S_i \to E$.



E.g., for Reach($\top$), **memoryless** strategies suffice to play optimally in all arenas.

Also suffice for Büchi, parity, mean-payoff, energy... objectives.

Memoryless determinacy

Memoryless determinacy

An objective $W \subseteq C^\omega$ is **memoryless-determined** if memoryless strategies suffice to play optimally for both players in all (finite) arenas.

We require *uniformity*: a **single** memoryless strategy must be winning from **all** the states where that is possible.

Memoryless determinacy

Good understanding of **memoryless determinacy** in **finite** arenas:

- **sufficient** conditions to guarantee memoryless optimal strategies for **both** players.[3,4]
- **sufficient** conditions to guarantee memoryless optimal strategies for **one** player.[5,6,7,8]
- **characterization of the objectives admitting memoryless optimal strategies for both players.**[9]

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[3] Gimbert and Zielonka, "When Can You Play Positionally?", 2004.

[4] Aminof and Rubin, "First-cycle games", 2017.

[5] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006.

[6] Gimbert, "Pure Stationary Optimal Strategies in Markov Decision Processes", 2007.

[7] Bianco et al., "Exploring the boundary of half-positionality", 2011.

[8] Gimbert and Kelmendi, "Submixing and Shift-Invariant Stochastic Games", 2014.

[9] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

Gimbert and Zielonka's characterization

One-to-two-player memoryless lift (finite arenas)[10]

Let $W \subseteq C^\omega$ be an objective. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has a memoryless optimal strategy,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has a memoryless optimal strategy,

then both players have a memoryless optimal strategy in all **two-player** arenas.

Extremely useful in practice. Very easy to recover memoryless determinacy of, e.g., parity and **mean-payoff** games.

[10] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

Application: memoryless determinacy of **mean payoff**[11]

- Colors $C = \mathbb{Q}$. Objective $W \subseteq C^\omega$ (for $\mathcal{P}_1$):
  obtain a **mean payoff** (average color by transition) ≥ 0.

- In finite **one-player** arenas, simply **reach** and loop around the **simple cycle** with the **greatest** (for $\mathcal{P}_1$) or **smallest** (for $\mathcal{P}_2$) **mean payoff** $\rightsquigarrow$ memoryless strategy.

$\overset{[GZ05]}{\Longrightarrow}$ Memoryless strategies also suffice to play optimally in **two-player** arenas!

[11] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

What happens in **infinite arenas**?

Motivations

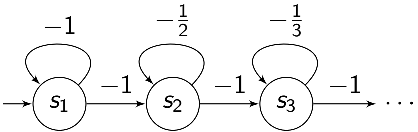
- Links between the **strategy complexity** in finite and **infinite** arenas?
- Similar sufficient conditions/characterizations in **infinite** arenas?

  $\rightsquigarrow$ Classical proof technique for finite arenas (induction on edges) not suited to infinite arenas.

Greater memory requirements in **infinite** arenas

Colors $C = \mathbb{Q}$, objective $W =$ "get a mean payoff ≥ 0".

- **Memoryless** strategies suffice in **finite** arenas.
- **Infinite** memory required in (even one-player) **infinite** arenas.[12]



$\rightsquigarrow$ Possible to get 0 at the limit **with infinite memory**:
loop increasingly many times in states s_n.

[12] Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, 1994.

Nice result in infinite arenas

Let $W \subseteq C^\omega$ be a **prefix-independent** objective.

Characterization of memoryless determinacy (infinite arenas)[13]

If **memoryless strategies** suffice to play optimally for both players in **infinite arenas**, then W is a **parity condition**.

Parity condition: there exists $p\colon C \to \{0, \dots, n\}$ such that

$$w = c_1 c_2 \ldots \in W \iff \limsup_i p(c_i) \text{ is even.}$$

Characterization since parity conditions are memoryless-determined.[14,15]

[13] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[14] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[15] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

Link with [GZ05]

Possible to obtain the result with a hypothesis on **one-player** arenas only!
Let $W \subseteq C^\omega$ be a **prefix-independent** objective.

Characterization of memoryless determinacy (infinite arenas)

If **memoryless strategies** suffice to play optimally for both players in **one-player infinite arenas**, then W is a **parity condition**.

Proof of **one-to-two-player lift**:

Memoryless determinacy in **one-player** infinite arenas

$\xrightarrow{[CN06]}$ W is a parity condition

$\Longrightarrow$ memoryless determinacy in **two-player** infinite arenas.

Two possible extensions

1. What about **strategies** with **finite memory**?
   $\rightsquigarrow$ More and more prevalent in the literature.

2. Some simple memoryless-determined objectives are not **prefix-independent** (e.g., $\mathsf{Reach}(\top)$).
   $\rightsquigarrow$ This characterization **misses** memoryless-determined objectives.

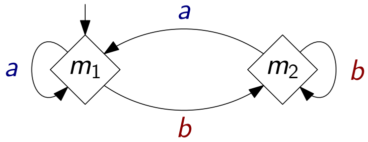
1. Finite memory

Finite-memory strategy ≈ memory structure + next-action function.

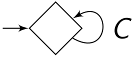
Memory structure

Memory structure $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$: finite set of states M, initial state m_{init}, update function $\alpha_{\mathsf{upd}} \colon M \times C \to M$.

Ex.: remember whether a or b was last played (**not yet a strategy!**):



Given an arena $\mathcal{A} = (S_1, S_2, E)$: *next-action function* $\alpha_{\mathsf{nxt}} \colon S_i \times M \to E$.

Memoryless strategies use **memory structure**  .

1. Finite-memory determinacy

Finite-memory determinacy

An objective W is **finite-memory-determined** if **there exists a finite memory structure** $\mathcal{M}$ that suffices to play optimally for both players **in all arenas** $\mathcal{A}$.

Technical comment

Usually, the definition inverts the order of the quantifiers. The order has an impact in **finite arenas**,[16] but not in **infinite arenas**.

[16] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2020.

2. Get rid of prefix-independence – Congruence

Let L be a language of **finite** words on alphabet C.

Right congruence

For $x, y \in C^*$, $x \sim_L y$ if for all $z \in C^*$, $xz \in L \Leftrightarrow yz \in L$.

Myhill-Nerode theorem[17]

- L is **regular** if and only if $\sim_L$ has **finitely many equivalence classes**.
- The **equivalence classes** of $\sim_L$ correspond to the **states of the minimal DFA for** L.

[17] Nerode, "Linear Automaton Transformations", 1958.

2. Get rid of prefix-independence – Congruence

Let W be a language of **infinite** words (= an objective) on alphabet C.

Right congruence

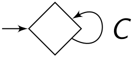
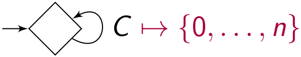
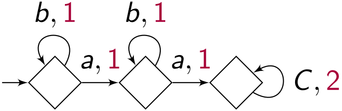
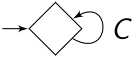
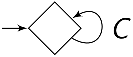
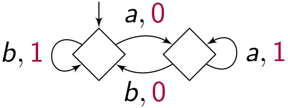
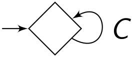
For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \Leftrightarrow yz \in W$.

Link with ω-regularity?

- If W is **ω-regular**, then $\sim_W$ has finitely many equivalence classes. In this case, there is a DFA $\mathcal{M}_\sim$ "prefix-classifier" associated with $\sim_W$.
- The reciprocal is not true.

W is prefix-independent if and only if $\sim_W$ has only one equivalence class.

Four examples

Objective	Prefix-classifier $\mathcal{M}_\sim$	Memory
$C = \{0, \ldots, n\}$, Parity condition		
$C = \{a, b\}$, $W = b^* a b^* a C^\omega$		
$C = \{a, b\}$, $W = C^*(ab)^\omega$		
$C = \mathbb{Q}$, $W = \mathsf{MP}^{\geq 0}$		No finite structure

Main result

Let $W \subseteq C^\omega$ be an objective.

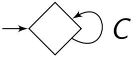
Theorem

If a finite memory structure $\mathcal{M}$ suffices to play optimally in **one-player** infinite arenas for both players, then

- ($\mathcal{M}_\sim$ is finite and)
- W **is recognized by a parity automaton** $(\mathcal{M}_\sim \otimes \mathcal{M}, p)$.

$\rightsquigarrow$ if $\mathcal{M}_\sim \otimes \mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$,

$$p \colon M \times C \to \{0, \dots, n\}.$$

Generalizes [CN06][18] ($\mathcal{M}_\sim = \mathcal{M} =$ 

).

[18] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Corollaries

Let $W \subseteq C^\omega$ be an objective.

One-to-two-player FM lift (infinite arenas)

If W is finite-memory-determined in **one-player** infinite arenas, then W is finite-memory-determined in **two-player** infinite arenas.

Characterization

W is **finite-memory-determined** if and only if W is ω-**regular**.

Proof. W is finite-memory-determined in **one-player** arenas
$\xRightarrow{[BRV22]}$ W is recognized by a deterministic parity automaton (ω-regular)
$\Longrightarrow$[19] this parity automaton (as a memory) suffices in **two-player** arenas
$\Longrightarrow$ this parity automaton (as a memory) suffices in **one-player** arenas.

[19] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

One-to-two-player lifts

When does two-player zero-sum memory determinacy reduce to **one-player** memory determinacy?

Arenas\Str. comp.	Memoryless	**FM "$\exists\mathcal{M}\forall\mathcal{A}$"**	Mildly growing
Finite deterministic	**[GZ05]**[20]	[BLORV20][21]	[Koz21][22]
Finite stochastic	[GZ09][23]	[BORV21][24]	
Infinite determin.	**P-Ind: [CN06]**[25]	**[BRV22]**[26]	

[20] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[21] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2020.

[22] Kozachinskiy, "One-to-Two-Player Lifting for Mildly Growing Memory", 2021.

[23] Gimbert and Zielonka, "Pure and Stationary Optimal Strategies in Perfect-Information Stochastic Games with Global Preferences", 2009.

[24] Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2021.

[25] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[26] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs", 2022.

Summary

Contributions

- New **one-to-two-player lift** for zero-sum games on infinite graphs.
- **Strategic characterization** of ω-regular languages. Strong link between representation as a DPA and memory structures.

Future work

- Only one player has FM optimal strategies?[27]
- How to find/compute minimal memory structures for synthesis?[28, 29]

Thanks!

[27] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2022.

[28] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

[29] Casares, "On the Minimisation of Transition-Based Rabin Automata and the Chromatic Memory Requirements of Muller Conditions", 2022.