Characterizing ω-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs

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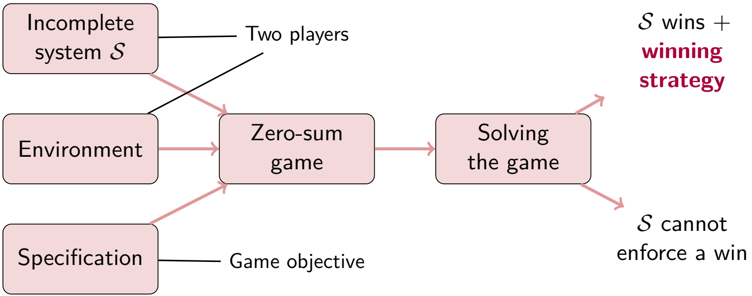
Context: **synthesis**

- A ***reactive* system** with some capabilities,
- living in an (uncontrollable) **environment**,
- with a purpose/**specification**.

$\rightsquigarrow$ Modeling through a *zero-sum game*.



Outline

Strategy synthesis for zero-sum games

Design **optimal** controllers for systems interacting with an **antagonistic** environment.

Interest in "simple" strategies

Finite-memory determinacy: when do **finite-memory** strategies suffice?

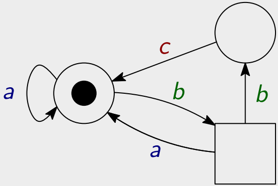
Inspiration

Results about **memoryless strategies**.[1]

[1] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Games

Zero-sum turn-based games on **graphs**



- **Colors** (events) C, **arena** $\mathcal{A} = (V_1, V_2, E)$.
- Two **players** $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$).
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.
- **Zero-sum**: objective of $\mathcal{P}_2$ is $C^\omega \setminus W$.

Strategies

A **strategy** of player $\mathcal{P}_i$ is a function $\sigma\colon E^* \to E$.
A strategy σ of $\mathcal{P}_1$ is **winning for** W **from** $v \in V$ if all infinite paths from v *consistent with* σ induce an infinite word in W.

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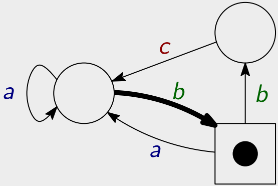

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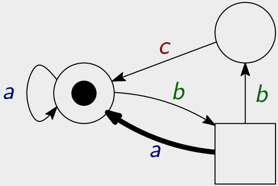

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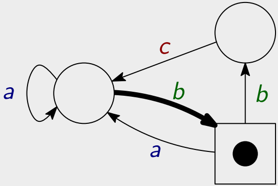

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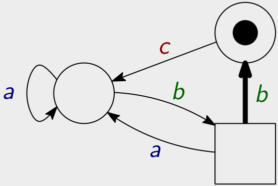

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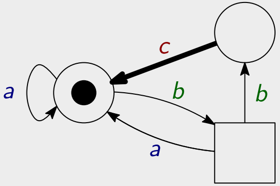

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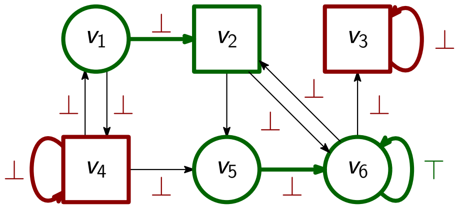
Memoryless strategies

Question

For an objective, do **simple** strategies suffice to win in all arenas (when winning is possible)?

A strategy σ of $\mathcal{P}_i$ is **memoryless** if it is a function ~~E^*~~ $V_i \to E$.

E.g., for $W = \text{Reach}(\top)$, **memoryless** strategies suffice to win.



In all arenas! Memoryless strategies also suffice for many other objectives.

Nice result

Let $W \subseteq C^\omega$ be a **prefix-independent** objective
(i.e., for all $w \in C^*$, $w' \in C^\omega$, we have $ww' \in W \Leftrightarrow w' \in W$).

Theorem [CN06][2]

If **memoryless strategies** suffice to win for **both** players in all **infinite arenas**, then W is a **parity condition**.

Parity condition: there exists $p\colon C \to \{0, \ldots, n\}$ such that

$$w = c_1 c_2 \ldots \in W \iff \limsup_i p(c_i) \text{ is even.}$$

Characterization (other implication was known).[3]

[2] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[3] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

Plan: Two possible extensions

1. What about **strategies** with **finite memory**?
   $\rightsquigarrow$ Already necessary for some simple and natural specifications.

2. Some simple memoryless-determined objectives are not **prefix-independent** (e.g., $\mathsf{Reach}(\top)$).
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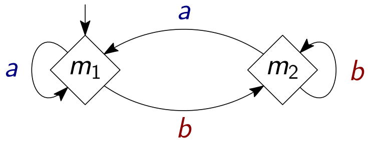
Finite memory

Finite-memory strategy $\approx$ memory structure + next-action function.

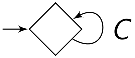
Memory structure

Memory structure $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$: finite set of states M, initial state m_{init}, update function $\alpha_{\mathsf{upd}}\colon M \times C \to M$.

Ex.: remember whether a or b was last seen:



Given an arena $\mathcal{A} = (V_1, V_2, E)$: *next-action function* $V_i \times M \to E$.

Memoryless strategies use **memory structure** 

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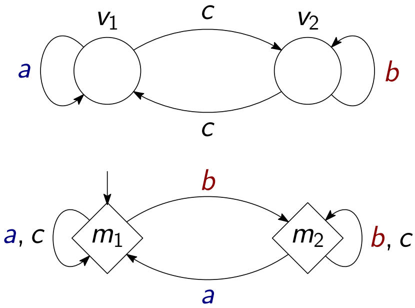
Example

$$C = \{a, b, c\},$$

$$W = \{w \in C^\omega \mid a \text{ is seen } \infty\text{ly often and } b \text{ is seen } \infty\text{ly often}\}$$



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but **two memory states** do! There is a winning strategy $\sigma \colon V_1 \times M \to E$.

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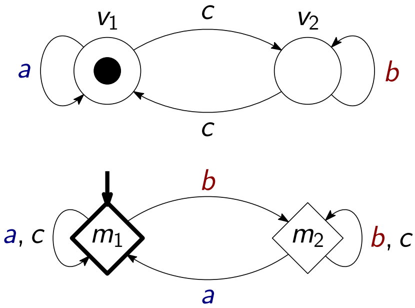

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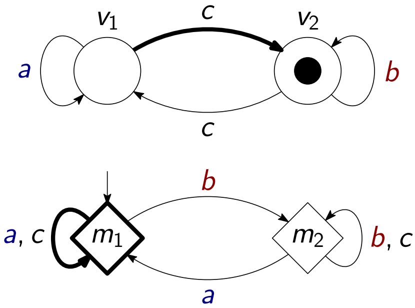

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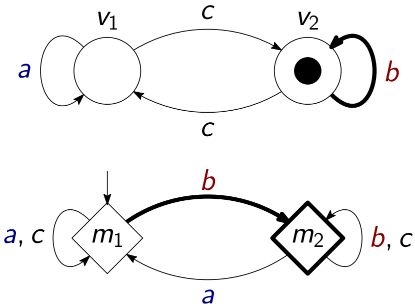

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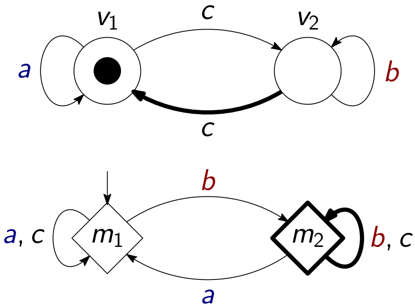

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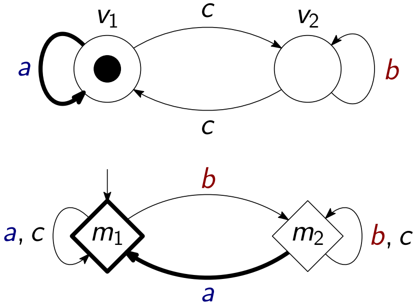

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Finite-memory determinacy

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Objective W is **finite-memory determined** if **there exists a finite memory structure** $\mathcal{M}$ that suffices to win for both players **in all arenas**.

Very useful property: if $\mathcal{M}$ is known and the arena is finite, only **finitely many** strategies to consider.

$\rightsquigarrow$ The winner of a game can be **decided**.

Remark

There are weaker definitions in which $\mathcal{M}$ may depend on the arena.[4]

[4] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2020.

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Get rid of prefix-independence?

Let L be a language of **finite** words on alphabet C.

Myhill-Nerode congruence

For $x, y \in C^*$, $x \sim_L y$ if for all $z \in C^*$, $xz \in L \Leftrightarrow yz \in L$.

Myhill-Nerode theorem[5]

L is **regular** if and only if $\sim_L$ has **finitely many equivalence classes**.
The **equivalence classes of** $\sim_L$ correspond to the **states of the minimal DFA for** L.

What about languages of **in**finite words and ω**-regularity**?

[5] Nerode, "Linear Automaton Transformations", 1958.

Get rid of prefix-independence?

Let W be a language of **infinite** words (= an objective) on alphabet C.

(Almost) Myhill-Nerode congruence

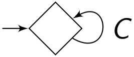
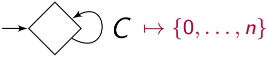
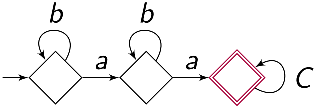
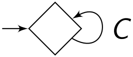
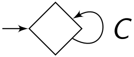
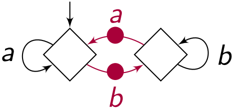
For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \Leftrightarrow yz \in W$.

Links with ω-regularity?

- If W is **ω-regular**, then $\sim_W$ has finitely many equivalence classes.
  $\rightsquigarrow$ Structure $\mathcal{M}_W$ "*prefix-classifier*" associated with $\sim_W$.
- Reciprocal not true.

W is **prefix-independent** if and only if $\sim_W$ has only one equivalence class.

Three examples

Objective	**Prefix-classifier $\mathcal{M}_W$**	**Memory $\mathcal{M}$**
$C = \{0, \ldots, n\}$, Parity condition		
$C = \{a, b\}$, $W = b^* a b^* a C^\omega$		
$W =$ "a and b ∞ly often"		

Main result

Let $W \subseteq C^\omega$ be an objective.

Theorem [Bouyer, Randour, V., 2022][6]

If a finite memory structure $\mathcal{M}$ suffices to win in infinite arenas for both players, **then** W **is recognized by a parity automaton** $(\mathcal{M}_W \otimes \mathcal{M}, p)$.

Generalizes [CN06][7] (memoryless, prefix-independent case).

Corollary

W is **finite-memory determined** if and only if W is ω-**regular**.

Direction "$\Leftarrow$" is a classical result.[8]

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[6] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs", 2022.

[7] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[8] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969; Rabin, "Decidability of Second-Order Theories and Automata on Infinite Trees", 1969.

Summary

Contributions

- **Strategic characterization** of ω-regularity, generalizing [CN06].[9]
- Strengthens **link** between the representation of an objective and its memory requirements.

Future work

- Precise memory requirements for each player given an objective?
- Similar characterizations in other game models (stochastic, concurrent, imperfect information. . .).

Thanks!

[9] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.