Characterizing ω-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs

Patricia Bouyer[1], Mickael Randour[2], Pierre Vandenhove[1,2]

[1]Université Paris-Saclay, CNRS, ENS Paris-Saclay, LMF, France

[2]F.R.S.-FNRS & UMONS – Université de Mons, Belgium

August 26, 2022 – LAMAS & SR 2022

Laboratoire Méthodes Formelles

FREEDOM TO RESEARCH

UMONS Université de Mons

Outline

Strategy synthesis for zero-sum turn-based games

Design **optimal** controllers for systems interacting with an **antagonistic** environment.

Interest in "simple" strategies

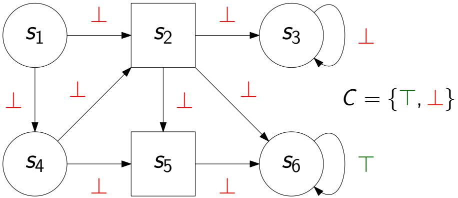
Finite-memory determinacy: when do **finite-memory** strategies suffice?
Focus on games on **infinite** graphs.

Inspiration

Results about **memoryless** **determinacy**.[1]

[1] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Zero-sum turn-based games on graphs



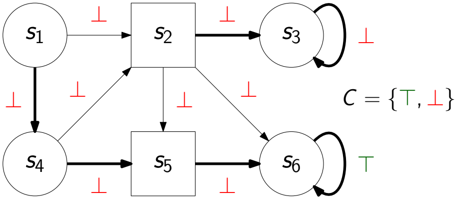
- Two-player **arenas**: S_1 ($\bigcirc$, for $\mathcal{P}_1$) and S_2 ($\square$, for $\mathcal{P}_2$), edges E.
- Set C of **colors**. **Edges** are colored.
- **Objectives** are sets $W \subseteq C^\omega$. **Zero-sum**.
- **Strategy** for $\mathcal{P}_i$: function $\sigma\colon E^* \to E$.

Memoryless determinacy

Question

For an objective, do *simple* strategies suffice to play optimally in all arenas?

A strategy σ of $\mathcal{P}_i$ is *memoryless* if it is a function ~~E^*~~ $S_i \to E$.



E.g., for Reach($\top$), **memoryless** strategies suffice to play optimally.

Also suffice for Büchi, parity... objectives.

Memoryless determinacy

Good understanding of **memoryless determinacy in finite arenas**

Sufficient conditions and **characterizations** of memoryless determinacy

- for **one** player,[2,3,4,5]
- for **both** players.[6,7,8]

What about **infinite** arenas?

[2] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006.

[3] Gimbert, "Pure Stationary Optimal Strategies in Markov Decision Processes", 2007.

[4] Bianco et al., "Exploring the boundary of half-positionality", 2011.

[5] Gimbert and Kelmendi, "Submixing and Shift-Invariant Stochastic Games", 2014.

[6] Gimbert and Zielonka, "When Can You Play Positionally?", 2004.

[7] Aminof and Rubin, "First-cycle games", 2017.

[8] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

What about **infinite arenas**?

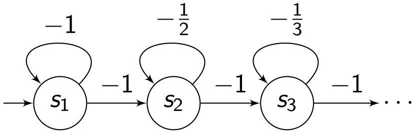
Motivations

- Links between the **strategy complexity** in finite and **infinite** arenas?
- Similar sufficient conditions/characterizations for **infinite** arenas?
  $\rightsquigarrow$ Classical proof technique for finite arenas (induction on number of edges) not suited to infinite arenas.

Greater memory requirements in **infinite** arenas

Colors $C = \mathbb{Q}$, objective W = "get a *mean payoff* ≥ 0".

- **Memoryless** strategies sufficient in **finite** arenas.[9]
- **Infinite** memory required in **infinite** arenas.[10]



$\rightsquigarrow$ Possible to get 0 at the limit **with infinite memory**:
loop increasingly many times in states s_n.

[9] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

[10] Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, 1994.

Nice result

Let $W \subseteq C^\omega$ be a **prefix-independent** objective.

Characterization of memoryless determinacy[11]

If **memoryless strategies** suffice to play optimally for **both** players in all **infinite arenas**, then W is a **parity condition**.

Parity condition: there exists $p\colon C \to \{0, \ldots, n\}$ such that

$$w = c_1 c_2 \ldots \in W \iff \limsup_i p(c_i) \text{ is even.}$$

Characterization since parity conditions are memoryless-determined.[12]

[11] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[12] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

Two possible extensions

1. What about **strategies** with **finite memory**?
   $\rightsquigarrow$ More and more prevalent in the literature.

2. Some simple memoryless-determined objectives are not **prefix-independent** (e.g., $\text{Reach}(\top)$).
   $\rightsquigarrow$ This characterization **misses** memoryless-determined objectives.

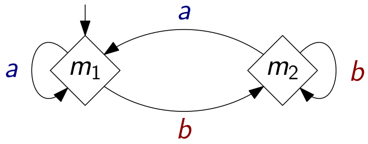
1. Finite memory

Finite-memory strategy $\approx$ memory structure + next-action function.

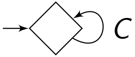
Memory structure

Memory structure $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$: finite set of states M, initial state m_{init}, update function $\alpha_{\mathsf{upd}} \colon M \times C \to M$.

Ex.: remember whether a or b was last seen:



Given an arena $\mathcal{A} = (S, S_1, S_2, E)$: *next-action function* $\alpha_{\mathsf{nxt}} \colon S_i \times M \to E$.

Memoryless strategies use **memory structure** 

.

1. Finite-memory determinacy

Finite-memory determinacy

An objective W is **finite-memory-determined** if **there exists** **a finite memory structure** $\mathcal{M}$ that suffices to play optimally for both players **in all arenas** $\mathcal{A}$.

Remark

Usually, the definition inverts the order of the quantifiers. The order has an impact in **finite arenas**,[13] but not in **infinite arenas**.

[13] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2020.

2. Get rid of prefix-independence? Congruence

Let L be a language of **finite** words on alphabet C.

Myhill-Nerode congruence

For $x, y \in C^*$, $x \sim_L y$ if for all $z \in C^*$, $xz \in L \Leftrightarrow yz \in L$.

Myhill-Nerode theorem[14]

L is **regular** if and only if $\sim_L$ has **finitely many equivalence classes**.
The **equivalence classes of** $\sim_L$ correspond to the **states of the minimal DFA for** L.

[14] Nerode, "Linear Automaton Transformations", 1958.

2. Get rid of prefix-independence? Congruence

Let W be a language of **infinite** words ($=$ an objective) on alphabet C.

Right congruence

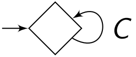
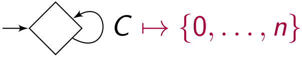
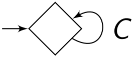
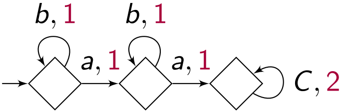
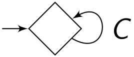
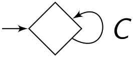
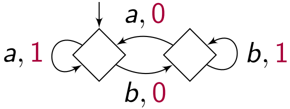
For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \Leftrightarrow yz \in W$.

Links with ω-regularity?

- If W is **ω-regular**, then $\sim_W$ has finitely many equivalence classes.
  In this case, there is a DFA $\mathcal{M}_\sim$ "*prefix-classifier*" associated with $\sim_W$.
- Reciprocal not true.

W is **prefix-independent** if and only if $\sim_W$ has only one equivalence class.

Four examples

<table>
<tr><th>Objective</th><th>Prefix-classifier $\mathcal{M}_\sim$</th><th>Memory</th></tr>
<tr><td>$C = \{0, \dots, n\}$,
Parity condition</td><td></td><td></td></tr>
<tr><td>$C = \mathbb{Q}$,
$W = \mathrm{MP}^{\geq 0}$</td><td></td><td>No finite structure</td></tr>
<tr><td>$C = \{a, b\}$,
$W = b^* a b^* a C^\omega$</td><td></td><td></td></tr>
<tr><td>$C = \{a, b\}$,
$W = C^*(ab)^\omega$</td><td></td><td></td></tr>
</table>

Main result

Let $W \subseteq C^\omega$ be an objective.

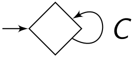
Theorem

If a finite memory structure $\mathcal{M}$ suffices to play optimally in infinite arenas for both players, **then**

- ($\mathcal{M}_\sim$ is finite, and)
- W **is recognized by a parity automaton** $(\mathcal{M}_\sim \otimes \mathcal{M}, p)$.

$\rightsquigarrow$ if $\mathcal{M}_\sim \otimes \mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$,

$$p \colon M \times C \to \{0, \ldots, n\}.$$

Generalizes [CN06][15] (where $\mathcal{M}_\sim = \mathcal{M} =$ 

).

[15] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Corollary

Let $W \subseteq C^\omega$ be an objective.

Characterization

W is **finite-memory-determined** if and only if W is ω-**regular**.

Proof. W is finite-memory-determined.
$\xRightarrow{[BRV22]}$ W is recognized by a deterministic parity automaton (ω-regular).
$\Longrightarrow$[16] this parity automaton (as a memory) suffices in infinite arenas.

[16] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

Summary

Contributions

- **Strategic characterization** of ω-regularity, generalizing [CN06].[17]
- (*Not mentioned*) New **one-to-two-player lift** for zero-sum games on infinite graphs.

Future work

- Other classes of arenas (e.g., finitely branching)?
- Only **one** player has finite-memory optimal strategies?[18]

Thanks!

[17] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[18] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2022.