Strategy Complexity of Zero-Sum Games on Graphs

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Overview

Strategy complexity

Understand if **complex** strategies must be used, or if **simple** strategies suffice to achieve an **objective** in presence of an **antagonistic** environment.

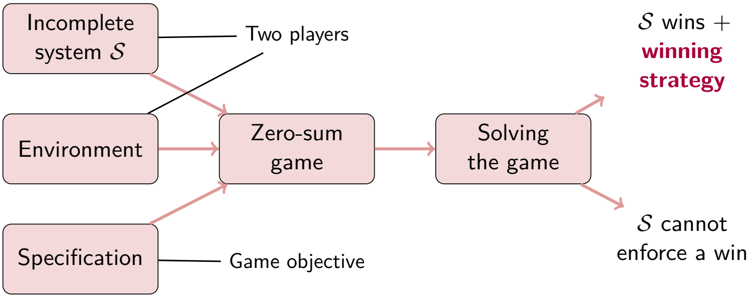
Aim of the talk

- **Motivate** the question of strategy complexity.
- Present the **state of the art**.
- Review **recent results** on the topic (part of my PhD thesis).

Context: **synthesis**

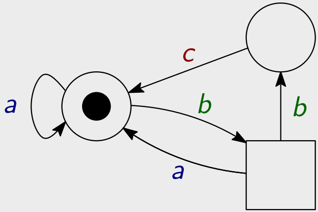
- An (incomplete, *reactive*) **system**,
- living in an (uncontrollable) **environment**,
- with a purpose/**specification**.

$\rightsquigarrow$ Modeling through a *zero-sum game*.



Games

Zero-sum turn-based games on **graphs**



- **Colors** C, **arena** $\mathcal{A} = (V_1, V_2, E)$.
- Two **players** $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\Box$).
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.
- **Zero-sum**: objective of $\mathcal{P}_2$ is $C^\omega \setminus W$.

Strategies

A **strategy** of player $\mathcal{P}_i$ is a function $\sigma \colon E^* \to E$.
A strategy σ of $\mathcal{P}_1$ is **winning for** W **from** $v \in V$ if all infinite paths from v *consistent with* σ induce an infinite word in W.

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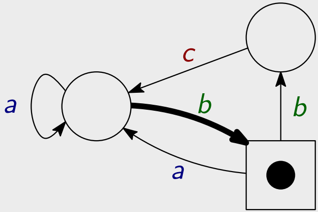

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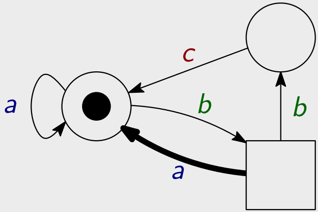

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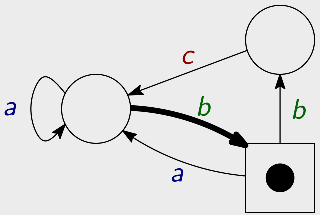

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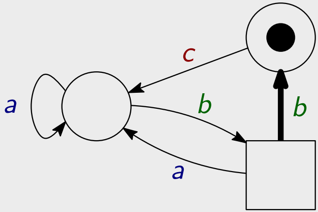

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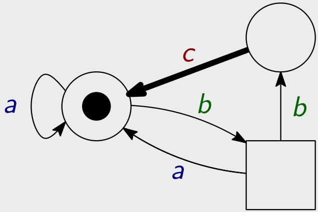

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Example

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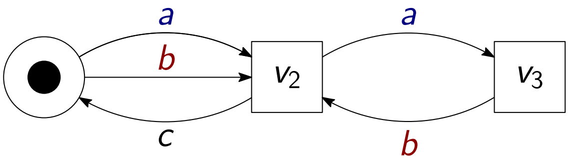

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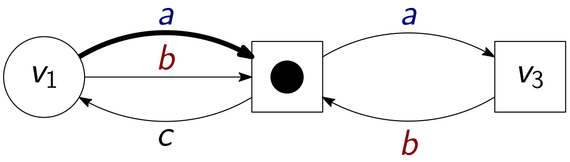

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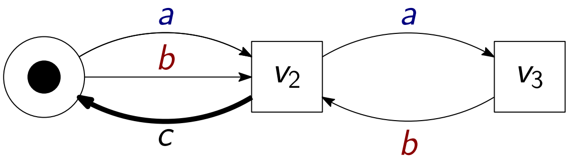

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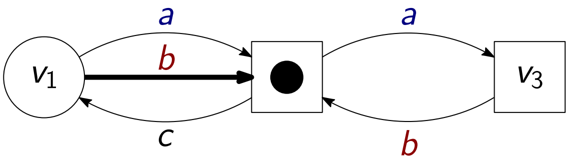

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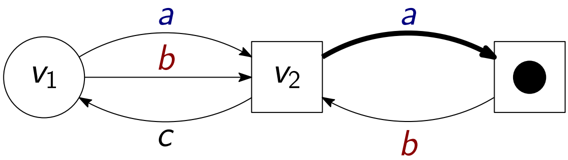

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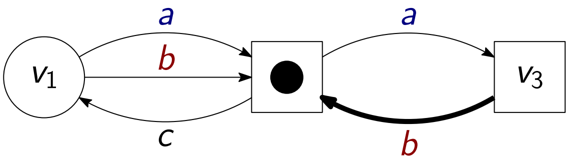

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Strategy complexity

- Given a game and an initial vertex $\rightsquigarrow$ **who can win?**
- To decide it, exhibit a **winning strategy** of a player.
- **Issues**:
  - strategies $\sigma \colon E^* \to E$ may not have a finite representation;
  - there are infinitely many of them.

Strategy complexity

Given an **objective**, understand if **simple** strategies suffice to win, or if **complex** strategies are required to win (when possible).

Desirable properties:

- winning strategies use bounded information (finite representation!);
- computable bounds (finite number of strategies!).

Simple strategies

Memoryless strategies

A strategy is **memoryless** if it makes decisions based only on the **current arena vertex** ($\sigma \colon V_i \to E$).

Finite-memory strategies

A strategy is **finite-memory** if it makes decisions based on

- the current arena vertex, **and**
- the current state of a finite *memory structure*.

Memory structures

Memory structures

A **memory structure** is a tuple $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ where M is a finite set of states, $m_{\mathsf{init}} \in M$, and $\alpha_{\mathsf{upd}} \colon M \times C \to M$.

Given $\mathcal{M}$ and an arena $\mathcal{A} = (V_1, V_2, E)$, a *finite-memory strategy* of $\mathcal{P}_i$ is a function

$$\sigma \colon V_i \times M \to E.$$

Remark

Memory structures are **chromatic**: only observe colors.
Given $\mathcal{A} = (V_1, V_2, E)$, slightly more general[1] to have

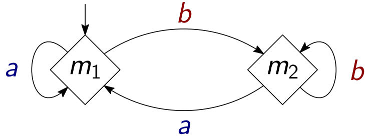
$$\alpha_{\mathsf{upd}} \colon M \times E \to M.$$

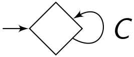
Still, we consider here **chromatic** structures (additional motivation later).

[1] Casares, "On the Minimisation of Transition-Based Rabin Automata and the Chromatic Memory Requirements of Muller Conditions", 2022.

Examples

E.g., structure to remember whether a or b was last seen:



Memoryless strategies use **memory structure** 

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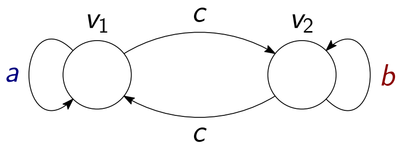

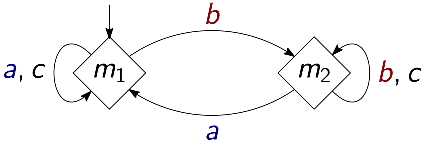

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but **two memory states** do! There is a winning strategy $\sigma \colon V_1 \times M \to E$.

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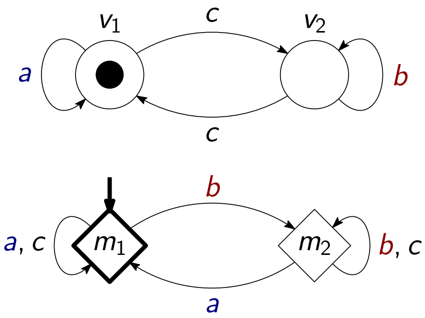

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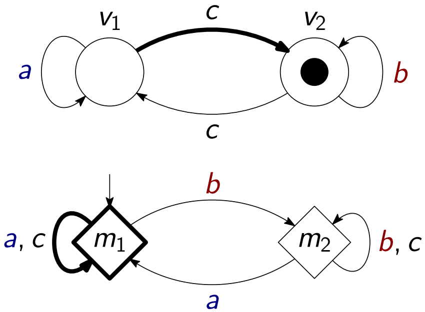

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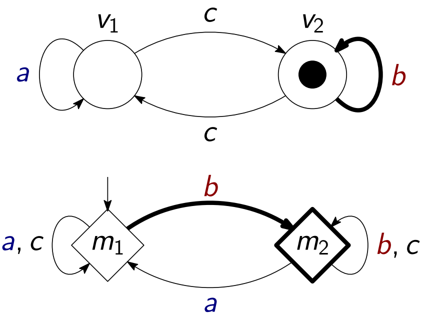

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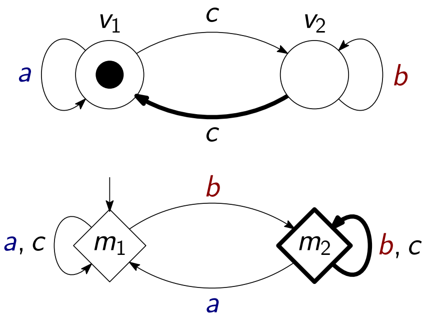

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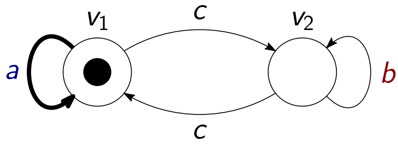

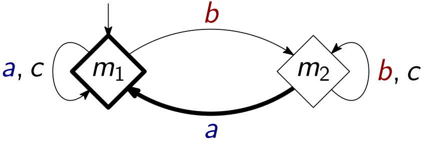

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Finite-memory determinacy

Memoryless determinacy

An **objective** is **memoryless-determined** if **in all arenas**, **memoryless** strategies suffice **for both players**.

Finite-memory determinacy

An **objective** is **finite-memory determined** if **in all arenas**, **finite-memory** strategies suffice **for both players**.

Various definitions depending on

- the class of **arenas** considered (finite, infinite, finitely branching...),
- whether we focus on **both** players or a **single** player.

State of the art: Memoryless determinacy

Many "classical" objectives are **memoryless-determined**:
reachability, Büchi, parity, energy, mean payoff, discounted sum...

Memoryless determinacy is well-understood:

- **Sufficient conditions** for **both** players,[2] for **a single** player.[3]
- **Characterizations** for **both** players over finite[4]/infinite[5] arenas, for **a single** player over infinite arenas.[6]

[2] Gimbert and Zielonka, "When Can You Play Positionally?", 2004; Aminof and Rubin, "First-cycle games", 2017.

[3] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006; Bianco et al., "Exploring the boundary of half-positionality", 2011.

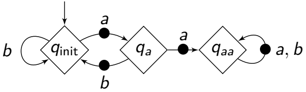
[4] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[5] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[6] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2023.

State of the art: Finite-memory determinacy

- **Finite-memory determinacy** is understood for specific objectives,[7] but few results of wide applicability.[8]
- Central class: **ω-regular objectives**. Examples with $C = \{a, b\}$:

ω-regular expressions	ω-automata	Linear temporal logic (LTL)
$b^* a b^* a C^\omega$		$\mathbf{GF}a$

Theorem[9, 10]

All **ω-regular objectives** are finite-memory determined.

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[7] Chatterjee, Randour, and Raskin, "Strategy synthesis for multi-dimensional quantitative objectives", 2014; Colcombet, Fijalkow, and Horn, "Playing Safe", 2014; Bouyer, Hofman, et al., "Bounding Average-Energy Games", 2017.

[8] Le Roux, Pauly, and Randour, "Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions", 2018; Bouyer, Le Roux, and Thomasset, "Finite-Memory Strategies in Two-Player Infinite Games", 2022.

[9] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[10] Rabin, "Decidability of Second-Order Theories and Automata on Infinite Trees", 1969.

Significance

Consequences of a fine-grained understanding of **strategy complexity**:

- **Decidability** of logical theories through FM det. (see *monadic second-order logic*, linked to ω-regular objectives[11]).
- Practical **synthesis** problems through FM det. (see, e.g., *LTL* specifications[12]).
- At the core of algorithms to **solve** games (see, e.g., *parity games*[13]).
- Controllers as **compact** as possible.

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[11] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[12] Pnueli, "The Temporal Logic of Programs", 1977.

[13] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

Plan: three results

- Two "theoretical" **characterizations** of finite-memory determinacy:
  - I. Reduction to **simpler arenas**.
  - II. Link with **automaton representation**.
- **III**. One "effective" characterization to **compute** small memory structures; focus on **ω-regular objectives**.

Joint works with P. Bouyer, A. Casares, N. Fijalkow, S. Le Roux, Y. Oualhadj, M. Randour.

I. Reduction to **simpler arenas**

Reduction to simpler arenas

One-to-two-player memoryless lift (**finite** arenas)[14]

Let $W \subseteq C^\omega$ be an objective. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has memoryless winning strategies,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has memoryless winning strategies,

then both players have memoryless winning strategies in **two-player** arenas.

Strategy complexity does not increase when an opponent is added!
Easy to recover **memoryless determinacy** of, e.g., **parity**[15] and **mean-payoff**[16] objectives.

What about **finite-memory determinacy**?

[14] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[15] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[16] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

What about finite-memory determinacy?

- **Counterexample** to a one-to-two-player lift for FM determinacy ☹.
- In the counterexample, **the size of the memory depends** on the size of the one-player **arenas**. **Motivates the restriction to**...

Arena-independent finite memory

An objective is *arena-independent finite-memory determined* if

there exists a memory structure $\mathcal{M}$ such that for all arenas $\mathcal{A}$,

strategies based on $\mathcal{M}$ suffice to win in $\mathcal{A}$.

- Requires **chromatic** memory structures.
- Still holds for ω-regular objectives!
- **One-to-two-player lift works**!

Example

$$C = \{a, b, c\},$$
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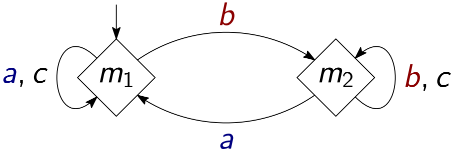
This memory structure actually suffices in **all** arenas!



$\rightsquigarrow$ W is **arena-independent** finite-memory determined.

One-to-two-player finite-memory lift

One-to-two-player FM lift [Bouyer, Le Roux, Oualhadj, Randour, V., 2022][17]

Let $W \subseteq C^\omega$ be an objective and $\mathcal{M}_1, \mathcal{M}_2$ be memory structures. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has winning strategies based on $\mathcal{M}_1$,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has winning strategies based on $\mathcal{M}_2$,

then both players have winning strategies based on $\mathcal{M}_1 \otimes \mathcal{M}_2$ in **two-player** arenas.

[17] Bouyer, Le Roux, Oualhadj, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

One-to-two-player lifts

When does strategy complexity in two-player zero-sum games
reduce to strategy complexity in **one-player** games?

Arenas \ Str. comp.	Memoryless	**FM "$\exists\mathcal{M}\forall\mathcal{A}$"**	Mildly growing
Finite	[GZ05][18]	[BLORV22][19]	[Koz22][20]
Infinite	[CN06][21]	[BRV23][22]	
Finite *stochastic*	[GZ09][23]	[BORV21][24]	

[18] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[19] Bouyer, Le Roux, Oualhadj, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

[20] Kozachinskiy, "One-To-Two-Player Lifting for Mildly Growing Memory", 2022.

[21] For prefix-independent objectives; Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[22] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

[23] Gimbert and Zielonka, "Pure and Stationary Optimal Strategies in Perfect-Information Stochastic Games with Global Preferences", 2009.

[24] Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2021.

II. Link with **automaton representation**

Second reduction: Link with automaton representation

Let $W \subseteq C^\omega$ be an objective.

(Almost) Myhill-Nerode congruence

For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \iff yz \in W$.

I.e., x and y have the same winning continuations; as good as each other.

Properties

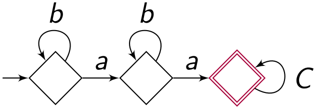
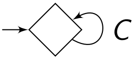
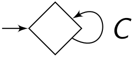
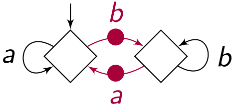
- If W is **ω-regular**, then $\sim_W$ has finitely many equivalence classes.
- There is a DFA $\mathcal{S}_W$ "**prefix classifier**" associated with $\sim_W$.

$\mathcal{S}_W$ might not "recognize" the objective ($\neq$ languages of *finite* words)...

Two examples

…but there is a *decomposition* with **prefix classifier** $\times$ **memory structure**.

Let $C = \{a, b\}$.

Objective	**Prefix classifier $\mathcal{S}_W$**	**Sufficient memory**
$W = b^* a b^* a C^\omega$		
$W =$ "a and b ∞ly often"		

From memory to automaton

Let $W \subseteq C^\omega$ be an objective.

Theorem [Bouyer, Randour, V., 2023][25]

If a finite memory structure $\mathcal{M}$ suffices to win in **infinite** arenas for both players, then

W **is recognized by a** ***parity automaton*** $(\mathcal{S}_W \otimes \mathcal{M}, p)$.

In particular,

W is **arena-independent finite-memory determined** over infinite arenas

$$\iff$$

W is ω-**regular**.

Generalizes [CN06][26] (prefix-independent, memoryless case).

[25] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

[26] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Up to now

Summary

Two characterizations to help study kinds of finite-memory determinacy.

Limits

Few hypotheses, but...

- not fully **effective**;
- in general, no tight memory requirements for **each** player.

III. Effective characterization

First: a well-studied case

For $\mathcal{F} \subseteq 2^C$, *objective Muller*$(\mathcal{F})$ is the set of words whose **set of colors seen infinitely often** is in $\mathcal{F}$.

Examples with $C = \{a, b\}$:

- $\text{Muller}(\{\{a\}, \{a, b\}\}) =$ "∞ly many a",
- $\text{Muller}(\{\{a, b\}\}) =$ "∞ly many a and ∞ly many b".

Memory requirements of Muller objectives

Series of papers between 1982 and 1998,[27,28,29,30] ending with a precise **characterization** and an **algorithm**.[31]

$\rightsquigarrow$ **Upper bound** on memory requirements for all ω-regular objectives!

[27] Gurevich and Harrington, "Trees, Automata, and Games", 1982.

[28] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[29] Klarlund, "Progress Measures, Immediate Determinacy, and a Subset Construction for Tree Automata", 1994.

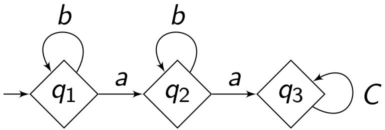
[30] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

[31] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

Why an upper bound?

Let $C = \{a, b\}$, $W = b^* a b^* a C^\omega$ ($\approx$ *seeing a two or more times*).
How to use results about **Muller objectives**?

W is not directly an objective Muller($\mathcal{F}$) with $\mathcal{F} \subseteq 2^C$
$\rightsquigarrow$ needs an **automaton structure**.



$\rightsquigarrow$ $W = \text{Muller}(\{\{q_3\}\})$.

Using [DJW97],[32] we need 1 memory state...

... **after** augmenting the arenas with the automaton,

so **upper bound of 3 states of memory**.

But 1 **memory state** suffices for winning strategies!

[32] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

Other direction: Regular objectives

Missing pieces

Alternative quest: objectives where "**finite prefixes matter**".

Regular objectives

- A **regular reachability objective** is a set LC^ω with $L \subseteq C^*$ regular.
- A **regular safety objective** is a set $C^\omega \setminus LC^\omega$.

Expressible as standard **deterministic finite automata**.

Question

Memory requirements of regular objectives

Characterize **the memory structures** that suffice to make optimal decisions for regular objectives **in any arena**. Compute **minimal** ones.

Ideas

- A **DFA recognizing the language** L, taken as a memory structure, always suffices for both players
  ($\approx$ usual approach: taking the product of the arena and the DFA).
- But can be much **smaller** in general!
- Properties linked to the Myhill-Nerode congruence.

I explain one of these properties here.

Comparing words

Let $W \subseteq C^\omega$ be an objective.

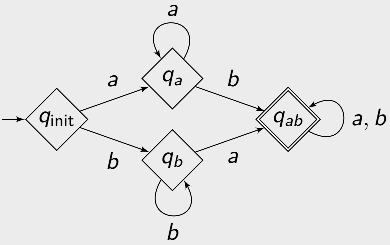
Comparing prefixes

For $x, y \in C^*$, $x \preceq_W y$ if for all $z \in C^\omega$, $xz \in W \implies yz \in W$.

I.e., y has more winning continuations than x; better situation.

Example

Let W be the regular **reachability** objective induced by this DFA.



E.g., $\varepsilon \prec_W a$,
a and b are *incomparable* for $\preceq_W$.

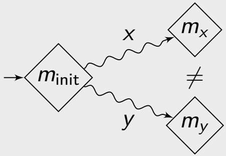
Necessary condition

Let $W \subseteq C^\omega$ be an objective, $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ be a memory structure.

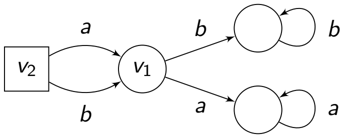
Lemma

For $\mathcal{M}$ to suffice for $\mathcal{P}_1$, $\mathcal{M}$ needs to **distinguish incomparable words**:

if $x, y \in C^*$ are incomparable for $\preceq_W$, then



Why? We can build an arena in which distinguishing x and y is critical.



Characterizations

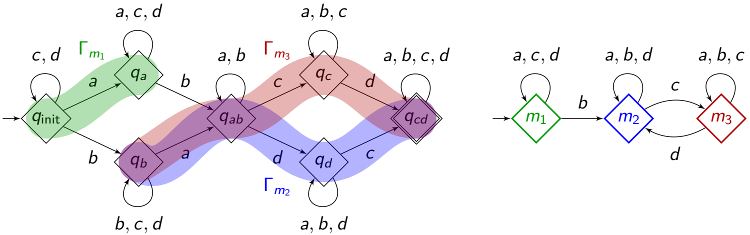
Theorem [Bouyer, Fijalkow, Randour, V., 2023][33]

Let W be a **regular safety objective**.

A memory structure $\mathcal{M}$ suffices in all arenas for $\mathcal{P}_1$

if and only if

$\mathcal{M}$ distinguishes incomparable words.



Close characterization for **regular reachability objectives**.

[33] Bouyer, Fijalkow, et al., "How to Play Optimally for Regular Objectives?", 2023.

Computational complexity

Decision problems

Input: An automaton $\mathcal{D}$ inducing the regular **reachability** (or **safety**) objective W and $k \in \mathbb{N}$.
Question: $\exists$ a memory structure $\mathcal{M}$ with $\leq k$ states that suffices for W?

Thanks to the "effectiveness" of the properties, we showed that:

Theorem[34]

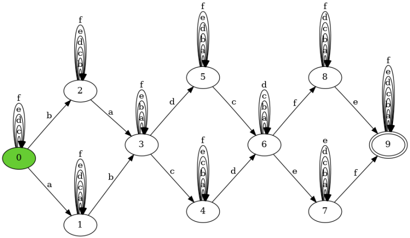
These problems are NP-complete.

[34] Bouyer, Fijalkow, et al., "How to Play Optimally for Regular Objectives?", 2022.

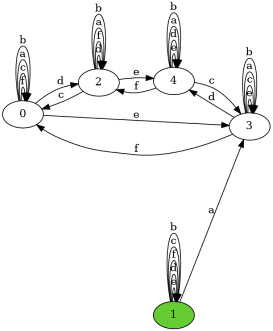
Implementation

Algorithms[35] that find minimal memory structures for regular objectives, using a **SAT solver**.

D =



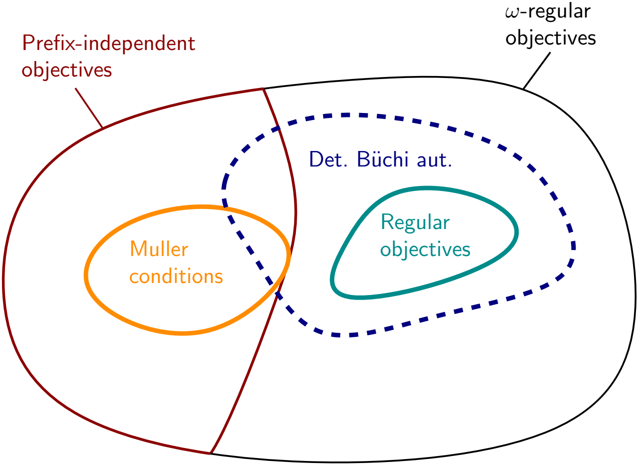
M = memReq.smallest_memory_safety(D) ⟶



[35] https://github.com/pvdhove/regularMemoryRequirements

Overview

Objectives with algorithms to compute **minimal** memory structures:



Only **memoryless** strategies for deterministic Büchi automata!

Future works

- Automatically **compute minimal memory structures** for *all* ω-regular objectives?
- More powerful **memory structures**.
  - Observing *edges* rather than colors (arena-*dependent*).
  - Well-behaved nondeterminism (*history-determinism*).[36]
- Practical advantage in knowing the minimal memory structure?

Thanks!

[36] Boker and Lehtinen, "When a Little Nondeterminism Goes a Long Way: An Introduction to History-Determinism", 2023.