Strategy Complexity of Zero-Sum Games on Graphs

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Thesis supervized by Patricia Bouyer[1] and Mickael Randour[2]

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March 14, 2023 – LMF Seminar

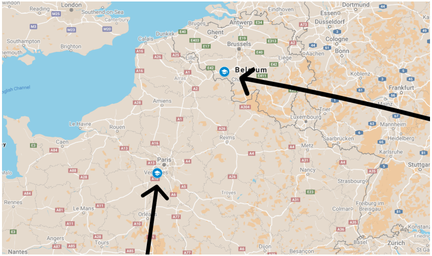
Laboratoire Méthodes Formelles

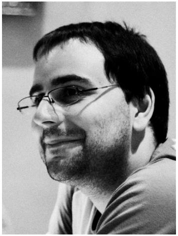
FREEDOM TO RESEARCH

UMONS Université de Mons

Context

- Present work part of my **thesis**.
- Thesis **supervised** by. . .





Mickael Randour,
Université de Mons,
Belgium

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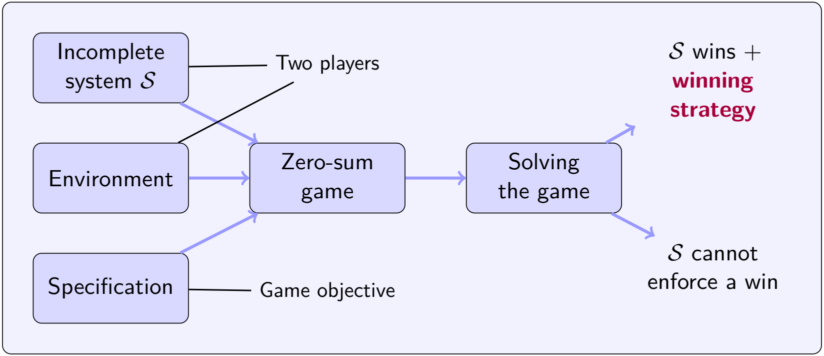
- **Thesis defense** in Mons at the end of April.

Problem: **synthesis**

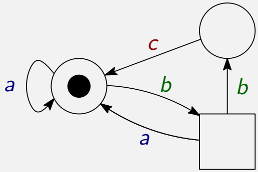
- An (incomplete, *reactive*) **system**,
- living in an (uncontrollable) **environment**,
- with a purpose/**specification**.

$\rightsquigarrow$ Modeling through a *zero-sum game*.



Zero-sum games

Zero-sum turn-based games on **graphs**



- **Colors** $C = \{a, b, c\}$, **arena** $\mathcal{A} = (V_1, V_2, E)$.
- Two **players** $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$).
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.
- **Zero-sum**: objective of $\mathcal{P}_2$ is $C^\omega \setminus W$.

Strategies

A **strategy** of a player is a function $\sigma \colon E^* \to E$.
A strategy σ of $\mathcal{P}_1$ is **winning for** W **from** $v \in V$ if all infinite paths from v *consistent with* σ induce an infinite word in W.

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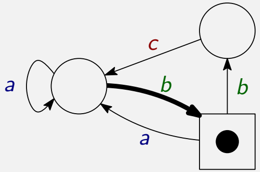

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  $\rightsquigarrow$ **infinite word** $w = b$
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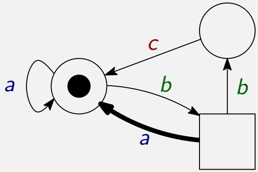

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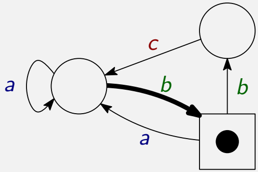

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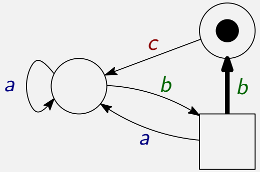

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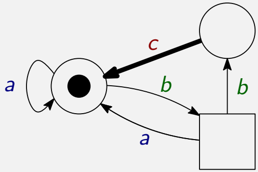

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Strategy complexity

- Given a game and an initial vertex $\rightsquigarrow$ **who can win?**
- To decide it, exhibit a **winning strategy** of a player.
- **Issues**:
  - strategies $\sigma \colon E^* \to E$ may not have a finite representation;
  - there are infinitely many of them.

Strategy complexity

Given an **objective**, *when winning is possible*, understand if **simple** strategies suffice to win, or if **complex** strategies are required.

Desirable properties:

- winning strategies can use bounded information (finite representation!);
- computable bounds on this information (finite number of strategies!).

Simple strategies

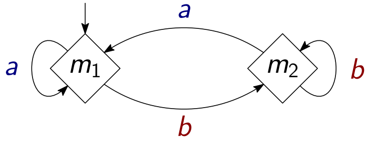
Memoryless strategies

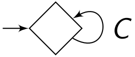
A strategy is **memoryless** if it makes decisions based only on the **current arena vertex** ($\sigma \colon V_i \to E$).

Finite-memory strategies

A strategy is **finite-memory** if it makes decisions based on the current arena vertex **and** the current state of a *memory structure* ($\sigma \colon V_i \times M \to E$).

Finite *memory structure* $\mathcal{M} = (M, m_{\text{init}} \in M, \alpha_{\text{upd}} \colon M \times C \to M)$.
E.g., to remember whether a or b was last played:



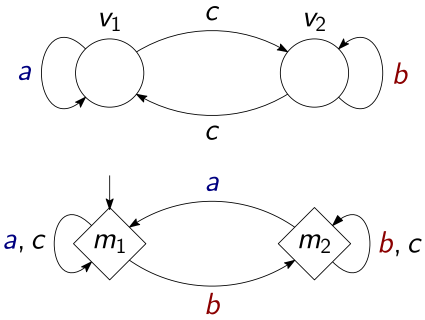
Memoryless strategies use **memory structure** 

.

Example

$$C = \{a, b, c\},$$

$$W = \{w \in C^\omega \mid a \text{ is seen } \infty\text{ly often and } b \text{ is seen } \infty\text{ly often}\}$$



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$\rightsquigarrow$ Memoryless strategies do **not** suffice...
but **two memory states** do!

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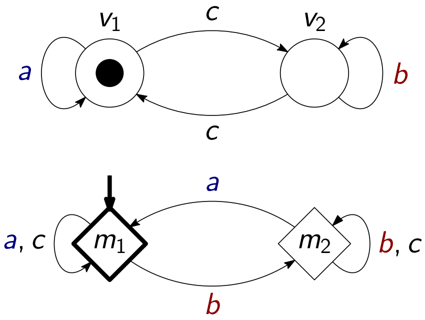

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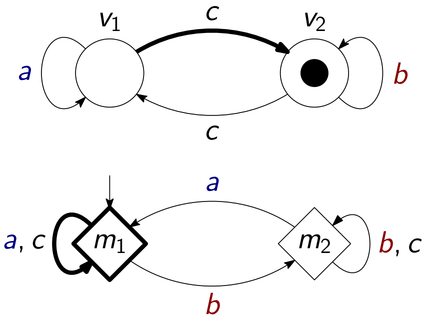

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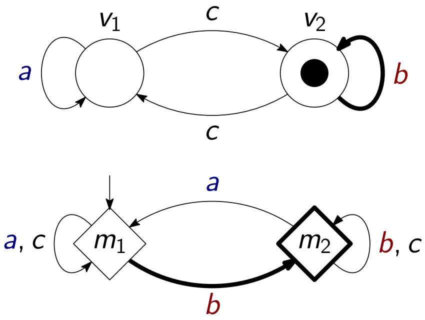

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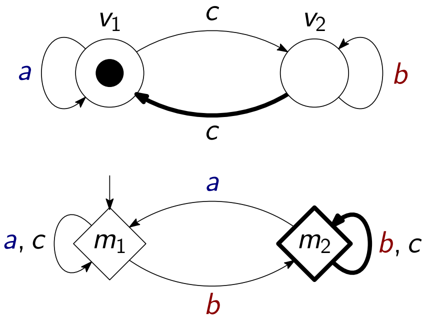

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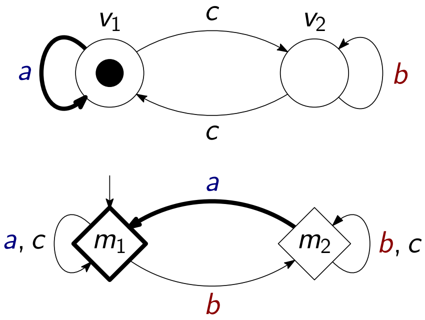

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⇝ Memoryless strategies do **not** suffice. . .
but **two memory states** do!

Finite-memory determinacy

Memoryless determinacy

An **objective** is **memoryless-determined** if **in all arenas**, **memoryless** strategies suffice **for both players**.

Finite-memory determinacy

An **objective** is **finite-memory-determined** if **in all arenas**, **finite-memory** strategies suffice **for both players**.

Various definitions depending on

- the class of **arenas** considered (finite, infinite, finitely branching...),
- whether we focus on **both** players or a **single** player.

State of the art: memoryless determinacy

Many "classical" objectives are **memoryless-determined**: reachability, Büchi, parity, energy, mean payoff, discounted sum...

Memoryless determinacy is well-understood:

- **Sufficient conditions** for **both** players,[1] for **a single** player.[2]
- **Characterizations** for **both** players over finite[3]/infinite[4] arenas, for **a single** player over infinite arenas.[5]

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[1] Gimbert and Zielonka, "When Can You Play Positionally?", 2004; Aminof and Rubin, "First-cycle games", 2017.

[2] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006; Bianco et al., "Exploring the boundary of half-positionality", 2011.

[3] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[4] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[5] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2022.

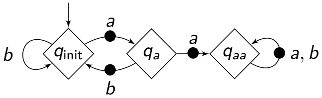
State of the art: finite-memory determinacy

- **Finite-memory determinacy** is understood for specific objectives,[6] but few results of wide applicability.[7]
- Central class: **ω-regular objectives**. Examples with $C = \{a, b\}$:

ω-regular expressions:

$b^* a b^* a C^\omega$

ω-automata:



Linear temporal logic (LTL):

$\mathbf{GF}a$

Theorem[8,9]

All ω-**regular objectives** are finite-memory-determined.

[6] Chatterjee, Randour, and Raskin, "Strategy synthesis for multi-dimensional quantitative objectives", 2014; Colcombet, Fijalkow, and Horn, "Playing Safe", 2014; Bouyer, Hofman, et al., "Bounding Average-Energy Games", 2017.

[7] Le Roux, Pauly, and Randour, "Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions", 2018; Bouyer, Le Roux, and Thomasset, "Finite-Memory Strategies in Two-Player Infinite Games", 2022.

[8] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[9] Rabin, "Decidability of Second-Order Theories and Automata on Infinite Trees", 1969.

Significance

Consequences of a fine-grained understanding of **strategy complexity**:

- **Decidability** of logical theories through FM det. (see *monadic second-order logic*, linked to ω-regular objectives).
- Practical **synthesis** problems through FM det. (see, e.g., *LTL* specifications[10]).
- At the core of algorithms to **solve** games (see, e.g., *parity games*[11]).
- Controllers as **compact** as possible.

[10] Pnueli, "The Temporal Logic of Programs", 1977.

[11] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

Overview of our contributions

I. General conditions for finite-memory determinacy

- **Arbitrary** objectives
- Algebraic **characterizations** of the sufficiency of a memory structure for both players
- **Theoretical tools** to find memory structures
- Generalizations of memoryless determinacy results

II. Precise memory requirements of classes of objectives

- ω-**regular** objectives
- **Observation**: memory requirements not settled

1. *Regular* objectives ($\approx$ DFAs)
   - Effective characterization of precise memory structures
   - (Computational) complexity
2. Objectives recognizable by *deterministic Büchi automata*
   - Effective characterization of "no memory for $\mathcal{P}_1$"
   - Complexity

I. General conditions for finite-memory determinacy

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- **Arbitrary** objectives
- Algebraic **characterizations** of the sufficiency of a memory structure for both players
- **Theoretical tools to help find memory structures**
  1. **One-to-two-player lifts**
  2. Memory structures
     $\rightsquigarrow$ automata for the objectives
- Generalizations of memoryless determinacy results

I. General conditions for finite-memory determinacy

1 One-to-two-player lift

***One-to-two-player* memoryless lift (finite arenas)**[12]

Let $W \subseteq C^\omega$ be an objective. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has memoryless winning strategies,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has memoryless winning strategies,

then both players have memoryless winning strategies in **two-player** arenas.

Extremely useful in practice. Very easy to recover **memoryless determinacy** of, e.g., **parity**[13] and **mean-payoff**[14] games.

What about **finite-memory determinacy**?

[12] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[13] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[14] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

1 What about finite-memory determinacy?

- **Counterexample** to a one-to-two-player lift for FM determinacy ☹.
- In the counterexample, **the size of the memory depends** on the size of the one-player **arenas**. **Motivates the restriction to**. . .

Arena-independent memory

An objective has *arena-independent finite-memory winning strategies* if

there exists a memory structure $\mathcal{M}$ such that for all arenas $\mathcal{A}$,

strategies using $\mathcal{M}$ suffice to win in $\mathcal{A}$.

- Still holds for ω-regular objectives!
- Restriction over finite arenas, not so much over infinite arenas.
- **One-to-two-player lift works**!

1 One-to-two-player lifts

When does memory determinacy in two-player zero-sum games
reduce to **one-player** memory determinacy?

Arenas\Str. comp.	Memoryless	**FM "$\exists\mathcal{M}\forall\mathcal{A}$"**	Mildly growing
Finite	**[GZ05]**[15]	**[BLORV22]**[16]	[Koz22][17]
Infinite	**[CN06]**[18]	**[BRV23]**[19]	
Finite stochastic	[GZ09][20]	**[BORV21]**[21]	

By-products of algebraic/language-theoretic characterizations.

[15] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[16] Bouyer, Le Roux, Oualhadj, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

[17] Kozachinskiy, "One-To-Two-Player Lifting for Mildly Growing Memory", 2022.

[18] For **prefix-independent** objectives; Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[19] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

[20] Gimbert and Zielonka, "Pure and Stationary Optimal Strategies in Perfect-Information Stochastic Games with Global Preferences", 2009.

[21] Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2021.

I. General conditions for finite-memory determinacy

- **Arbitrary** objectives
- Algebraic **characterizations** of the sufficiency of a memory structure for both players
- **Theoretical tools to help find memory structures**
  1. One-to-two-player lifts
  2. **Memory structures $\rightsquigarrow$ automata for the objectives**
- Generalizations of memoryless determinacy results

I. General conditions for finite-memory determinacy

2 Link with automaton representation

Let $W \subseteq C^\omega$ be an objective.

$\approx$ Myhill-Nerode congruence

For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \iff yz \in W$.

I.e., x and y have the same winning continuations; as good as each other.

Properties

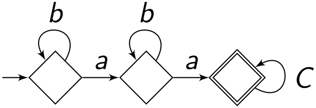
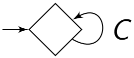
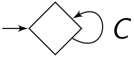
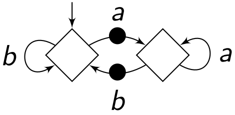
- If W is **ω-regular**, then $\sim_W$ has finitely many equivalence classes.
- There is a DFA $\mathcal{S}_W$ "prefix classifier" associated with $\sim_W$.

Might not "recognize" the language ($\neq$ languages of *finite* words)...

2 Two examples

...but we noticed a *decomposition* involving **prefix classifiers** and **memory structures**.

Let $C = \{a, b\}$.

Objective	**Prefix-classifier $\mathcal{S}_W$**	**Sufficient memory**
$W = b^* a b^* a C^\omega$		
$W =$ "a and b ∞ly often"		

Main result

Let $W \subseteq C^\omega$ be an objective.

Theorem

If a finite memory structure $\mathcal{M}$ suffices to play optimally in **infinite** arenas for both players, then

W **is recognized by a *parity automaton*** $(\mathcal{S}_W \otimes \mathcal{M}, p)$.

$\Longrightarrow W$ is ω-regular!

Generalizes [CN06][22] (prefix-independent, memoryless case).

[22] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

2 Corollary

Characterization

Let W be an objective.

$$W \text{ is } \textbf{finite-memory-determined} \text{ over infinite arenas} \iff W \text{ is } \omega\textbf{-regular}.$$

$\Longleftarrow$ is well-known.[23,24]

$\Longrightarrow$ follows from the previous slide.

[23] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[24] Rabin, "Decidability of Second-Order Theories and Automata on Infinite Trees", 1969.

Part I: Summary

- Useful **notion** of *arena-independent* FM determinacy.
- General **characterizations** over finite and infinite arenas.
- Theoretical **tools** to determine memory requirements.
- Central place of ω-**regular objectives**.

Related publications

- Bouyer, Le Roux, Oualhadj, Randour, V. (CONCUR'20 & LMCS) "Games Where You Can Play Optimally with Arena-Independent Finite Memory"
- Bouyer, Randour, V. (STACS'22 & TheoretiCS) "Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs"

Limits

Wide applicability, but. . .

- not fully **effective**;
- in general, no tight memory requirements for **each** player.

II. Precise memory requirements of classes of objectives

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- ω-**regular** objectives
- **Observation**: memory requirements not settled

1. ***Regular* objectives ($\approx$ DFAs)**
   - **Effective characterization of precise memory structures**
   - **Existence of small structures is NP-complete**
2. Objectives recognizable by *deterministic Büchi automata*
   - Effective characterization of "no memory for $\mathcal{P}_1$"
   - Decidable in polynomial time

1 Regular objectives

Well-understood ω-regular objectives: *Muller conditions*, focusing on what is seen **infinitely often**.[25,26]
E.g., $b^*ab^*aC^\omega$ is not a Muller condition.

Missing pieces

Orthogonal quest: objectives where "**finite prefixes matter**".

We consider the "simplest" ones.

Regular objectives

- A **regular reachability objective** is a set LC^ω with $L \subseteq C^*$ regular.
- A **regular safety objective** is a set $C^\omega \setminus LC^\omega$.

Expressible as standard **deterministic finite automata**.

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[25] Gurevich and Harrington, "Trees, Automata, and Games", 1982.

[26] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

1 Question

Memory requirements of regular objectives

Characterize the memory structures that suffice to make optimal decisions for regular objectives **in any arena**. Compute **minimal** ones.

Ideas

- A DFA recognizing the language, taken as a memory structure, always suffices for both players.
- But can be much smaller in general!
- Properties linked to the Myhill-Nerode congruence.

Comparing words

Let $W \subseteq C^\omega$ be an objective.

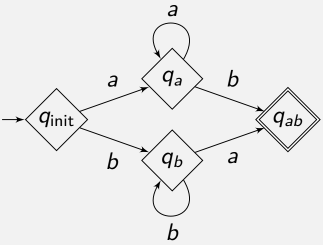
Comparing prefixes

For $x, y \in C^*$, $x \preceq_W y$ if for all $z \in C^\omega$, $xz \in W \implies yz \in W$.

I.e., y has more winning continuations than x; better situation.

Example

Let W be the regular **reachability** objective induced by this DFA.



E.g., $\varepsilon \prec_W a$, $a \prec_W ab$,
a and b are *incomparable* for $\preceq_W$.

Necessary condition for the memory

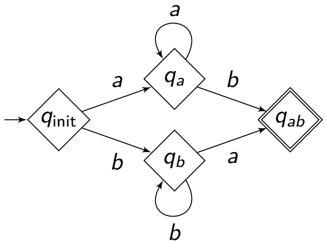
Let $W \subseteq C^\omega$ be an objective.

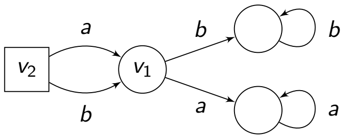
Lemma

A sufficient memory structure $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ needs to **distinguish incomparable words** (for $\preceq_W$), i.e.,

$$\text{if } x, y \in C^* \text{ are incomparable for } \preceq_W, \\ \text{then } \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, x) \neq \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, y).$$

Why? (Example) need to make the right decision in this arena.





1 Characterizations

Theorem

Let W be a **regular safety objective**.

A memory structure $\mathcal{M}$ suffices in all arenas for $\mathcal{P}_1$
if and only if
$\mathcal{M}$ distinguishes incomparable words.

Theorem

Let W be a **regular reachability objective**.

Memory structure $\mathcal{M}$ suffices in all arenas for $\mathcal{P}_1$
if and only if
$\mathcal{M}$ distinguishes incomparable words **and**
$\mathcal{M}$ *distinguishes insufficient progress*.

1 Computational complexity

Decision problems

Input: An automaton $\mathcal{D}$ inducing the regular **reachability** (or **safety**) objective W and $k \in \mathbb{N}$.
Question: $\exists$ a memory structure $\mathcal{M}$ with $\leq k$ states that suffices for W?

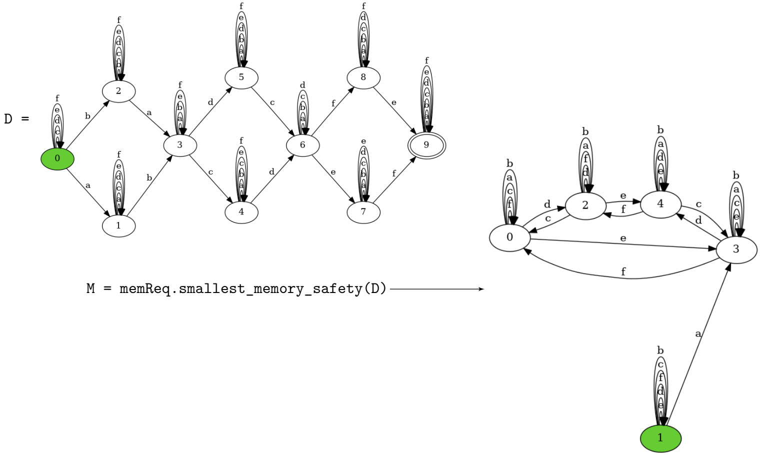
Thanks to the "effectiveness" of the two properties, we showed that:

Theorem

These problems are NP-complete.

1 Implementation

Algorithms that find minimal memory structures for regular objectives, using a **SAT solver**.



II. Precise memory requirements of classes of objectives

- ω-**regular** objectives
- **Observation**: memory requirements not settled

1. *Regular* objectives ($\approx$ DFAs)
   - Effective characterization of precise memory structures
   - Existence of small structures is NP-complete
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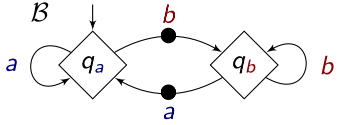
II. Precise memory requirements of classes of objectives

2 Deterministic Büchi automata

A **deterministic Büchi automaton** $\mathcal{B}$ on C

- reads **infinite** words (in C^ω),
- accepts words that see infinitely many **Büchi transitions** •.



$$\mathcal{L}(\mathcal{B}) = \{w \in \{a, b\}^\omega \mid w \text{ sees } \infty\text{ly many } a \text{ and } \infty\text{ly many } b\}$$

Question

Given $\mathcal{B}$, **can $\mathcal{P}_1$ win without memory for objective $W = \mathcal{L}(\mathcal{B})$**?
(Is $\mathcal{L}(\mathcal{B})$ *half-positional*?)

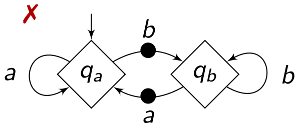
2 Results

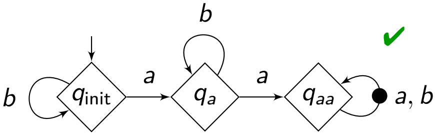
Let $\mathcal{B}$ be a **deterministic Büchi automaton**.

Theorem

For objective $W = \mathcal{L}(\mathcal{B})$, $\mathcal{P}_1$ does not need memory **if and only if**

- all prefixes are comparable for $\preceq_W$,
- W *is progress-consistent*, and
- W **is recognized by its prefix classifier as a DBA**.





Polynomial-time algorithm

Can be **decided** in $\mathcal{O}(|\mathcal{B}|^4)$ time.

Part II: Summary

- Tools to study memory req. of classes of ω-regular objectives.
- Effective **characterizations** for DFAs and DBAs.
- **Decidability** and **complexity** of the related decision problems.

Related publications

- Bouyer, Fijalkow, Randour, V. (Submitted preprint) "How to Play Optimally for Regular Objectives?"
- Bouyer, Casares, Randour, V. (CONCUR'22) "Half-Positional Objectives Recognized by Deterministic Büchi Automata"

Future works

- (*Part I*) General results for arena-*dependent* memory requirements.
  - Observing *edges* rather than colors in the model.
  - Well-behaved nondeterminism (*history-determinism*).[27]
- (*Part II*) Automatically **compute minimal memory structures** for *all* ω-regular objectives?
- More expressive **settings** (e.g., stochastic, concurrent,[28] or timed games).
- More expressive **strategy models** than finite-state machines (e.g., pushdown[29] or register[30] automata).

Thanks!

[27] Boker and Lehtinen, "When a Little Nondeterminism Goes a Long Way: An Introduction to History-Determinism", 2023.

[28] Bordais, Bouyer, and Le Roux, "Optimal Strategies in Concurrent Reachability Games", 2022.

[29] Walukiewicz, "Pushdown Processes: Games and Model-Checking", 2001.

[30] Exibard et al., "Computability of Data-Word Transductions over Different Data Domains", 2022.