# Strategy Complexity of Zero-Sum Games on Graphs

Pierre Vandenhove

LaBRI, Université de Bordeaux

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LABORATOIRE BORDELAIS DE RECHERCHE EN INFORMATIQUE LaBRI

# Overview

## Strategy complexity

Understand if **complex** strategies must be used, or if **simple** strategies suffice to achieve an **objective** in presence of an **antagonistic** environment.

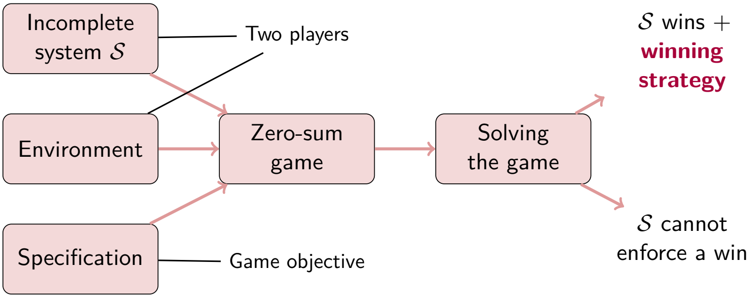
## Aim of the talk

- **Motivate** the question of strategy complexity.
- Present the **state of the art**.
- Review **recent results** on the topic (part of my PhD thesis).

# Context: **synthesis**

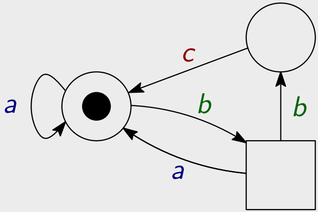
- An (incomplete, *reactive*) **system**,
- living in an (uncontrollable) **environment**,
- with a purpose/**specification**.

⇝ Modeling through a *zero-sum game*.



# Games

## Zero-sum turn-based games on **graphs**




- **Colors** $C$, **arena** $\mathcal{A} = (V_1, V_2, E)$.
- Two **players** $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$).
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.
- **Zero-sum**: objective of $\mathcal{P}_2$ is $C^\omega \setminus W$.

## Strategies

A **strategy** of player $\mathcal{P}_i$ is a function $\sigma\colon E^* \to E$.
A strategy $\sigma$ of $\mathcal{P}_1$ is **winning for** $W$ **from** $v \in V$ if all infinite paths from $v$ *consistent with* $\sigma$ induce an infinite word in $W$.

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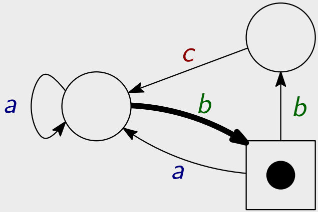


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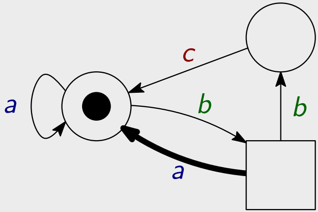


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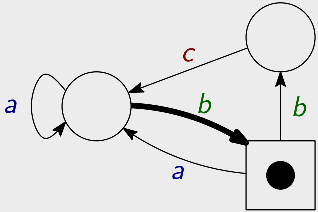


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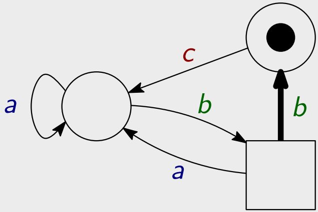


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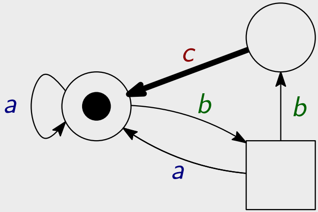


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# Example

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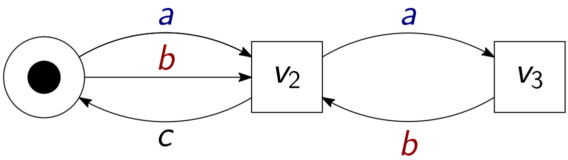


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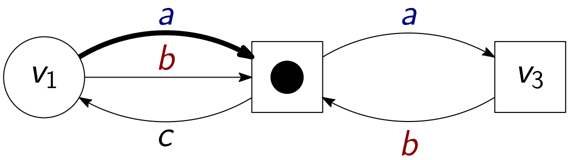


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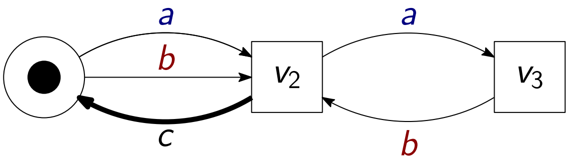


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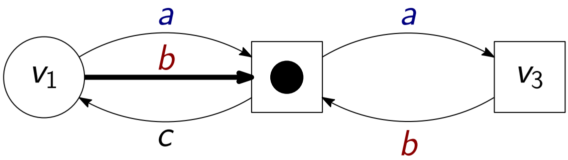


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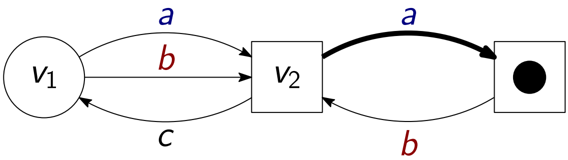


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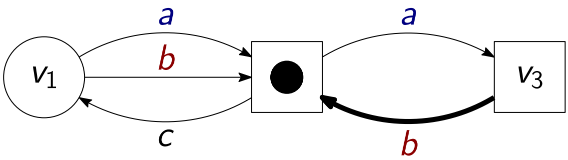


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# Strategy complexity

- Given a game and an initial vertex $\rightsquigarrow$ **who can win?**
- To decide it, exhibit a **winning strategy** of a player.
- **Issues**:
  - strategies $\sigma\colon E^* \to E$ may not have a finite representation;
  - there are infinitely many of them.

## Strategy complexity

Given an **objective**, understand if **simple** strategies suffice to win,
or if **complex** strategies are required to win (when possible).

Desirable properties:

- winning strategies use bounded information (finite representation!);
- computable bounds (finite number of strategies!).

# Simple strategies

## Memoryless strategies

A strategy is **memoryless** if it makes decisions based only on the **current arena vertex** ($\sigma\colon V_i \to E$).

## Finite-memory strategies

A strategy is **finite-memory** if it makes decisions based on

- the current arena vertex, **and**
- the current state of a finite *memory structure*.

# Memory structures

## Memory structures

A **memory structure** is a tuple $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ where $M$ is a finite set of states, $m_{\mathsf{init}} \in M$, and $\alpha_{\mathsf{upd}} \colon M \times C \to M$.

Given $\mathcal{M}$ and an arena $\mathcal{A} = (V_1, V_2, E)$, a *finite-memory strategy* of $\mathcal{P}_i$ is a function

$$\sigma \colon V_i \times M \to E.$$

## Remark

Memory structures are **chromatic**: only observe colors.
Given $\mathcal{A} = (V_1, V_2, E)$, slightly more general[1] to have

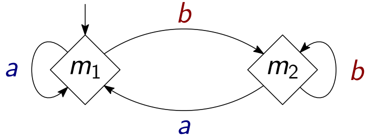
$$\alpha_{\mathsf{upd}} \colon M \times E \to M.$$

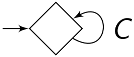
Still, we consider here **chromatic** structures (additional motivation later).

[1] Casares, "On the Minimisation of Transition-Based Rabin Automata and the Chromatic Memory Requirements of Muller Conditions", 2022.

# Examples

E.g., structure to remember whether $a$ or $b$ was last seen:




Memoryless strategies use **memory structure** 

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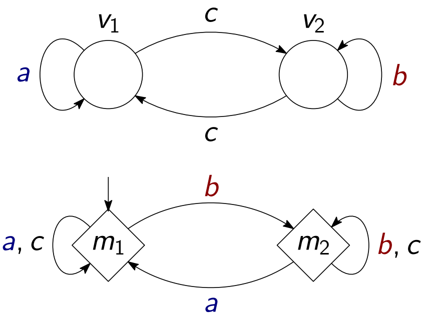


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⇝ Memoryless strategies do **not** suffice. . .
but **two memory states** do! There is a winning strategy $\sigma\colon V_1 \times M \to E$.

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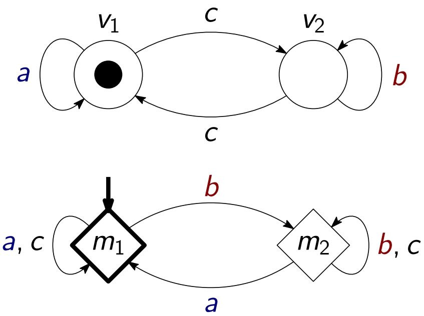


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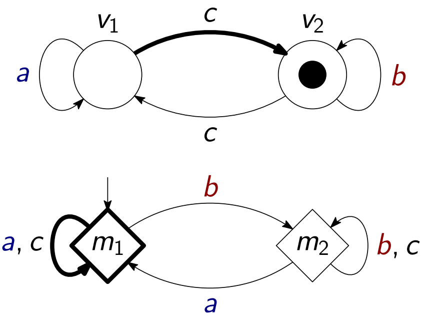


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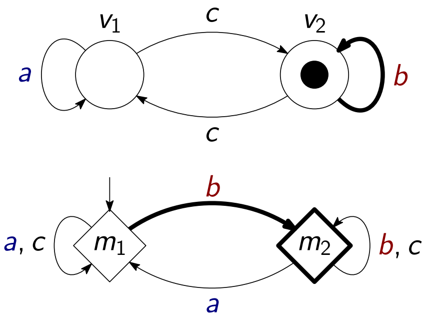


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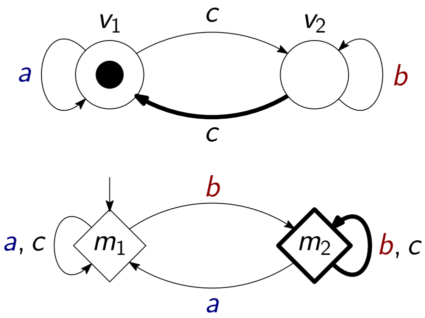


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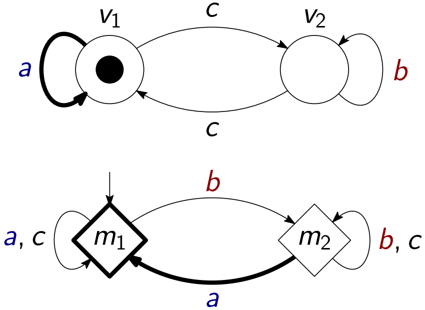


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# Finite-memory determinacy

## Memoryless determinacy

An **objective** is **memoryless-determined** if **in all arenas**, **memoryless** strategies suffice **for both players**.

## Finite-memory determinacy

An **objective** is **finite-memory determined** if **in all arenas**, **finite-memory** strategies suffice **for both players**.

Various definitions depending on

- the class of **arenas** considered (finite, infinite, finitely branching. . . ),
- whether we focus on **both** players or a **single** player.

# State of the art: Memoryless determinacy

Many "classical" objectives are **memoryless-determined**:
reachability, Büchi, parity, energy, mean payoff, discounted sum. . .

**Memoryless determinacy** is well-understood:

- **Sufficient conditions** for **both** players,[2] for **a single** player.[3]
- **Characterizations** for **both** players over finite[4]/infinite[5] arenas, for **a single** player over infinite arenas.[6]

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[2] Gimbert and Zielonka, "When Can You Play Positionally?", 2004; Aminof and Rubin, "First-cycle games", 2017.

[3] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006; Bianco et al., "Exploring the boundary of half-positionality", 2011.

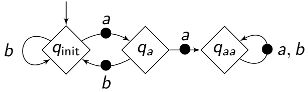
[4] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[5] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[6] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2023.

# State of the art: Finite-memory determinacy

- **Finite-memory determinacy** is understood for specific objectives,[7] but few results of wide applicability.[8]
- Central class: **$\omega$-regular objectives**. Examples with $C = \{a, b\}$:

| $\omega$-regular expressions | $\omega$-automata | Linear temporal logic (LTL) |
|---|---|---|
| $b^* a b^* a C^\omega$ |  | $\mathbf{GF}a$ |



**Theorem[9,10]**

All $\omega$-**regular objectives** are finite-memory determined.

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[7] Chatterjee, Randour, and Raskin, "Strategy synthesis for multi-dimensional quantitative objectives", 2014; Colcombet, Fijalkow, and Horn, "Playing Safe", 2014; Bouyer, Hofman, et al., "Bounding Average-Energy Games", 2017.

[8] Le Roux, Pauly, and Randour, "Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions", 2018; Bouyer, Le Roux, and Thomasset, "Finite-Memory Strategies in Two-Player Infinite Games", 2022.

[9] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[10] Rabin, "Decidability of Second-Order Theories and Automata on Infinite Trees", 1969.

# Significance

Consequences of a fine-grained understanding of **strategy complexity**:

- **Decidability** of logical theories through FM det. (see *monadic second-order logic*, linked to $\omega$-regular objectives[11]).
- Practical **synthesis** problems through FM det. (see, e.g., *LTL* specifications[12]).
- At the core of algorithms to **solve** games (see, e.g., *parity games*[13]).
- Controllers as **compact** as possible.

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[11] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[12] Pnueli, "The Temporal Logic of Programs", 1977.

[13] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

# Plan: three results

- Two "theoretical" **characterizations** of finite-memory determinacy:
  - I. Reduction to **simpler arenas**.
  - II. Link with **automaton representation**.
- **III**. One "effective" characterization to **compute** small memory structures; focus on **$\omega$-regular objectives**.

**Joint works** with P. Bouyer, A. Casares, N. Fijalkow, S. Le Roux, Y. Oualhadj, M. Randour.

# I. Reduction to **simpler arenas**

# Reduction to simpler arenas: existing result

*One-to-two-player* memoryless lift (**finite** arenas)[14]

Let $W \subseteq C^\omega$ be an objective. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has memoryless winning strategies,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has memoryless winning strategies,

then both players have memoryless winning strategies in **two-player** arenas.

Strategy complexity does not increase when an opponent is added!
Easy to recover **memoryless determinacy** of, e.g., **parity**[15] and **mean-payoff**[16] objectives.

What about **finite-memory determinacy**?

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[14] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[15] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[16] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

# What about finite-memory determinacy?

- **Counterexample** to a one-to-two-player lift for FM determinacy ☹.
- In the counterexample, **the size of the memory depends** on the size of the one-player **arenas**. **Motivates the restriction to**. . .

**Arena-independent** finite memory

An objective is *arena-independent finite-memory determined* if

**there exists a memory structure $\mathcal{M}$ such that for all arenas $\mathcal{A}$**,

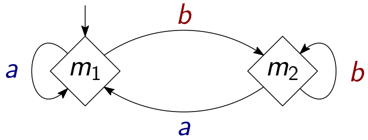
strategies based on $\mathcal{M}$ suffice to win in $\mathcal{A}$.

- Requires **chromatic** memory structures.
- Still holds for $\omega$-regular objectives!
- **One-to-two-player lift works**!

# Example

$$C = \{a, b, c\},$$
$$W = \{w \in C^\omega \mid a \text{ is seen } \infty\text{ly often and } b \text{ is seen } \infty\text{ly often}\}$$

This memory structure actually suffices in **all** arenas!




$\rightsquigarrow$ $W$ is **arena-independent** finite-memory determined.

# One-to-two-player finite-memory lift

**One-to-two-player FM lift** [Bouyer, Le Roux, Oualhadj, Randour, V., 2022][17]

Let $W \subseteq C^\omega$ be an objective and $\mathcal{M}_1, \mathcal{M}_2$ be memory structures. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has winning strategies based on $\mathcal{M}_1$,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has winning strategies based on $\mathcal{M}_2$,

then both players have winning strategies based on $\mathcal{M}_1 \otimes \mathcal{M}_2$ in **two-player** arenas.

[17] Bouyer, Le Roux, Oualhadj, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

# One-to-two-player lifts

**When does strategy complexity in two-player zero-sum games**
reduce to strategy complexity in **one-player** games?

| Arenas\Str. comp. | Memoryless | **FM "$\exists\mathcal{M}\forall\mathcal{A}$"** | Mildly growing |
|---|---|---|---|
| Finite | [GZ05][18] | [BLORV22][19] | [Koz22][20] |
| Infinite | [CN06][21] | [BRV23][22] | |
| Finite *stochastic* | [GZ09][23] | [BORV21][24] | |

[18] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[19] Bouyer, Le Roux, Oualhadj, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

[20] Kozachinskiy, "One-To-Two-Player Lifting for Mildly Growing Memory", 2022.

[21] For prefix-independent objectives; Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[22] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

[23] Gimbert and Zielonka, "Pure and Stationary Optimal Strategies in Perfect-Information Stochastic Games with Global Preferences", 2009.

[24] Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2021.

# II. Link with **automaton representation**

# Second reduction: Link with automaton representation

Let $W \subseteq C^\omega$ be an objective.

### (Almost) Myhill-Nerode congruence

For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \iff yz \in W$.

I.e., $x$ and $y$ have the same winning continuations; as good as each other.

### Properties

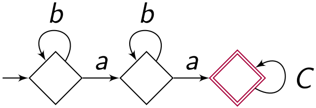
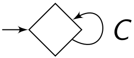
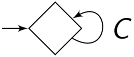
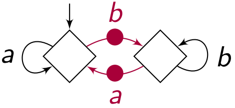
- If $W$ is **$\omega$-regular**, then $\sim_W$ has finitely many equivalence classes.
- There is a DFA $\mathcal{S}_W$ "**prefix classifier**" associated with $\sim_W$.

$\mathcal{S}_W$ might not "recognize" the objective ($\neq$ languages of *finite* words)...

# Two examples

. . . but there is a *decomposition* with **prefix classifier** $\times$ **memory structure**.

Let $C = \{a, b\}$.

| **Objective** | **Prefix classifier $\mathcal{S}_W$** | **Sufficient memory** |
|---|---|---|
| $W = b^* a b^* a C^\omega$ |   |   |
| $W =$ "$a$ and $b$ $\infty$ly often" |   |   |

# From memory to automaton

Let $W \subseteq C^\omega$ be an objective.

**Theorem** [Bouyer, Randour, V., 2023][25]

If a finite memory structure $\mathcal{M}$ suffices to win in **infinite** arenas for both players, then

$W$ **is recognized by a** ***parity automaton*** $(\mathcal{S}_W \otimes \mathcal{M}, p)$.

In particular,

$W$ is **arena-independent finite-memory determined** over infinite arenas

$\iff$

$W$ is $\omega$-**regular**.

Generalizes [CN06][26] (prefix-independent, memoryless case).

[25] Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

[26] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

# Up to now

## Summary

Two characterizations to help study kinds of finite-memory determinacy.

## Limits

Few hypotheses, but. . .

- not fully **effective**;
- in general, no tight memory requirements for **each** player.

# III. Effective characterization

# First: a well-studied case

For $\mathcal{F} \subseteq 2^C$, *objective Muller*$(\mathcal{F})$ is the set of words whose **set of colors seen infinitely often** is in $\mathcal{F}$.

Examples with $C = \{a, b\}$:

- $\mathrm{Muller}(\{\{a\}, \{a, b\}\})$ = "$\infty$ly many $a$",
- $\mathrm{Muller}(\{\{a, b\}\})$ = "$\infty$ly many $a$ and $\infty$ly many $b$".

## Memory requirements of Muller objectives

Series of papers between 1982 and 1998,[27,28,29,30] ending with a precise **characterization** and an **algorithm**.[31]

$\rightsquigarrow$ **Upper bound** on memory requirements for all $\omega$-regular objectives!

[27] Gurevich and Harrington, "Trees, Automata, and Games", 1982.

[28] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[29] Klarlund, "Progress Measures, Immediate Determinacy, and a Subset Construction for Tree Automata", 1994.

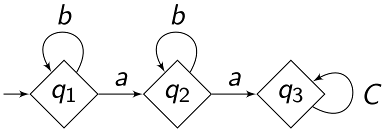
[30] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

[31] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

# Why an upper bound?

Let $C = \{a, b\}$, $W = b^*ab^*aC^\omega$ ($\approx$ *seeing a two or more times*).
How to use results about **Muller objectives**?

$W$ is not directly an objective $\text{Muller}(\mathcal{F})$ with $\mathcal{F} \subseteq 2^C$
$\rightsquigarrow$ needs an **automaton structure**.




$\rightsquigarrow W = \text{Muller}(\{\{q_3\}\})$.

Using [DJW97],[32] we need 1 memory state...
... **after** augmenting the arenas with the automaton,
so **upper bound of 3 states of memory**.

But **1 memory state** suffices for winning strategies!

[32] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

# Other direction: Regular objectives

## Missing pieces

Alternative quest: objectives where "**finite prefixes matter**".

We consider the "simplest" ones.

## Regular objectives

- A **regular reachability objective** is a set $LC^\omega$ with $L \subseteq C^*$ regular.
- A **regular safety objective** is a set $C^\omega \setminus LC^\omega$.

Expressible as standard **deterministic finite automata**.

# Question

## Memory requirements of regular objectives

**Characterize the memory structures** that suffice to make optimal decisions for regular objectives **in any arena**. Compute **minimal** ones.

## Ideas

- A **DFA recognizing the language** $L$, taken as a memory structure, always suffices for both players
  ($\approx$ usual approach: taking the product of the arena and the DFA).
- But can be much **smaller** in general!
- Properties linked to the Myhill-Nerode congruence.

I explain one of these properties here.

# Comparing words

Let $W \subseteq C^\omega$ be an objective.

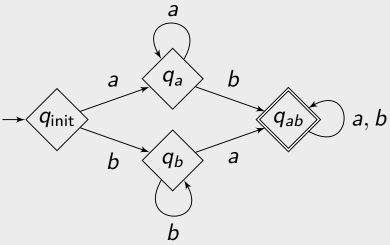
## Comparing prefixes

For $x, y \in C^*$, $x \preceq_W y$ if for all $z \in C^\omega$, $xz \in W \implies yz \in W$.

I.e., $y$ has more winning continuations than $x$; better situation.

## Example

Let $W$ be the regular **reachability** objective induced by this DFA.




E.g., $\varepsilon \prec_W a$,
$a$ and $b$ are *incomparable* for $\preceq_W$.

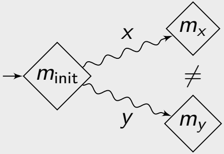
# Necessary condition

Let $W \subseteq C^\omega$ be an objective, $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ be a memory structure.

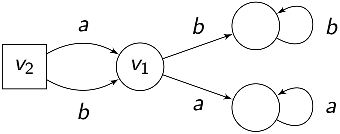
## Lemma

For $\mathcal{M}$ to suffice for $\mathcal{P}_1$, $\mathcal{M}$ needs to **distinguish incomparable words**:

if $x, y \in C^*$ are incomparable for $\preceq_W$, then




**Why**? We can build an arena in which distinguishing $x$ and $y$ is critical.



# Characterizations

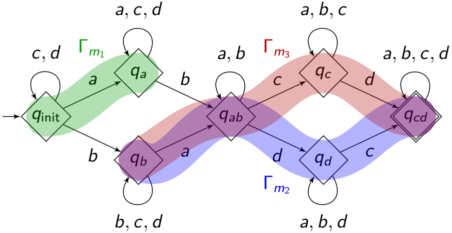
**Theorem** [Bouyer, Fijalkow, Randour, V., 2023][33]

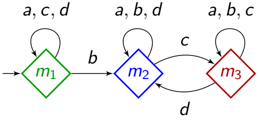
Let $W$ be a **regular safety objective**.

A memory structure $\mathcal{M}$ suffices in all arenas for $\mathcal{P}_1$

**if and only if**

$\mathcal{M}$ distinguishes incomparable words.




Close characterization for **regular reachability objectives**.

[33] Bouyer, Fijalkow, et al., "How to Play Optimally for Regular Objectives?", 2023.

# Computational complexity

### Decision problems

**Input**: An automaton $\mathcal{D}$ inducing the regular **reachability** (or **safety**) objective $W$ and $k \in \mathbb{N}$.
**Question**: $\exists$ a memory structure $\mathcal{M}$ with $\leq k$ states that suffices for $W$?

Thanks to the "effectiveness" of the properties, we showed that:

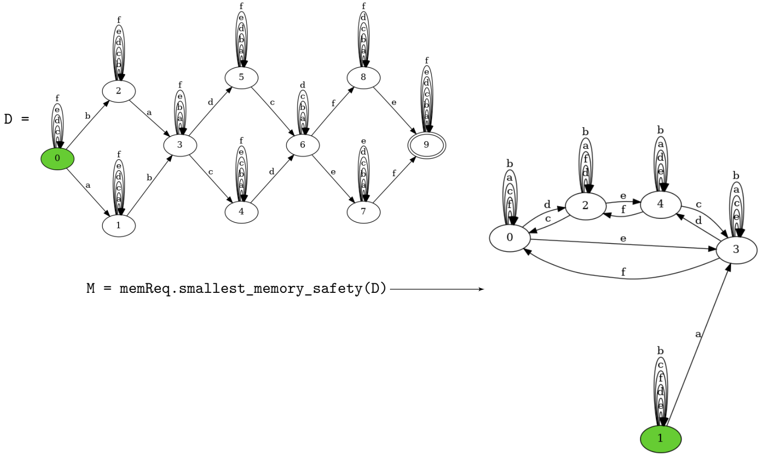
### Theorem[34]

These problems are NP-complete.

[34] Bouyer, Fijalkow, et al., "How to Play Optimally for Regular Objectives?", 2022.

# Implementation

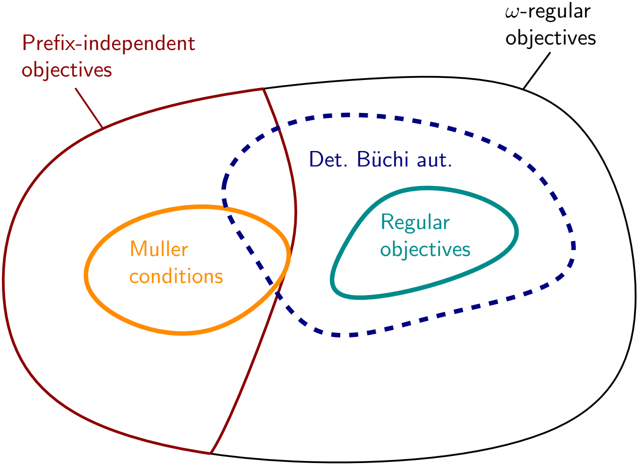
Algorithms[35] that find minimal memory structures for regular objectives, using a **SAT solver**.




[35] https://github.com/pvdhove/regularMemoryRequirements

# Overview

Objectives with algorithms to compute **minimal** memory structures:




Only memoryless strategies for deterministic Büchi automata!

# Future works

- More powerful **memory structures**.
  - Observing *edges* rather than colors (arena-*dependent*).
  - Well-behaved nondeterminism (*history-determinism*).[36]
- Automatically **compute minimal memory structures** for *all* $\omega$-regular objectives?
- Practical advantage in knowing the minimal memory structure?

# Thanks!

[36] Boker and Lehtinen, "When a Little Nondeterminism Goes a Long Way: An Introduction to History-Determinism", 2023.