Strategy Complexity of Zero-Sum Games on Graphs

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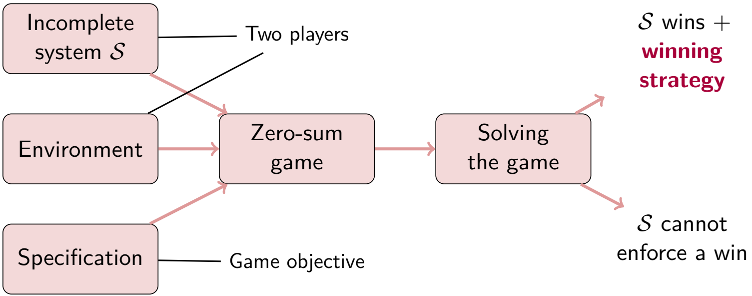
Context: **synthesis**

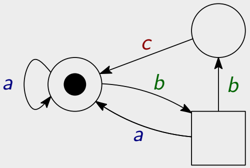
- An (incomplete, *reactive*) **system**,
- living in an (uncontrollable) **environment**,
- with a purpose/**specification**.

$\rightsquigarrow$ Modeling through a *zero-sum game*.



Games

Zero-sum turn-based games on **graphs**



- **Colors** C, edge-colored **arena** $\mathcal{A} = (V_1, V_2, E)$.
- Two **players** $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$).
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.
- **Zero-sum**: objective of $\mathcal{P}_2$ is $C^\omega \setminus W$.

Strategies

A **strategy** of player $\mathcal{P}_\ell$ is a function $\sigma\colon \mathsf{Hists}_\ell(\mathcal{A}) \to E$.
A strategy σ of $\mathcal{P}_1$ is **winning for W from $v \in V$** if all infinite paths from v *consistent with* σ induce an infinite word in W.

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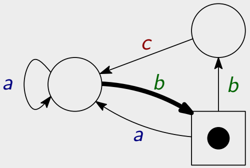

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  $\rightsquigarrow$ **infinite word** $w = b$
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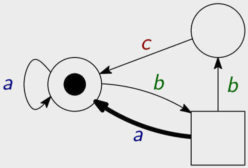

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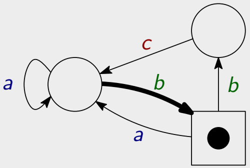

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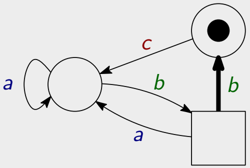

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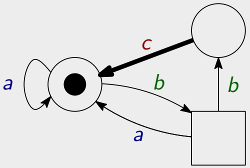

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Strategy complexity

- Given a game and an initial vertex $\rightsquigarrow$ **who can win?**
- To decide it, exhibit a **winning strategy** of a player.
- **Issues**:
  - strategies $\sigma\colon \mathsf{Hists}_\ell(\mathcal{A}) \to E$ may not have a finite representation;
  - there are infinitely many of them.

Strategy complexity

Given an **objective**, understand if **simple** strategies suffice to win, or if **complex** strategies are required to win (when possible).

Desirable properties:

- winning strategies use bounded information (finite representation!);
- computable bounds on this information (finite number of strategies!).

Simple strategies

Memoryless strategies

A strategy is **memoryless** if it makes decisions based only on the **current arena vertex** ($\sigma \colon V_\ell \to E$).

Finite-memory strategies

A strategy is **finite-memory** if it makes decisions based on

- the current arena vertex, **and**
- the current state of a finite *memory structure*.

Memory structures

Memory structures

Finite **memory structure** $\mathcal{M} = (M, m_{\mathsf{init}} \in M, \alpha_{\mathsf{upd}})$. Two kinds:

- Given $\mathcal{A} = (V_1, V_2, E)$, memory structure $\mathcal{M}$ (for $\mathcal{A}$) is **chaotic** if

$$\alpha_{\mathsf{upd}} \colon M \times E \to M.$$

- Memory structure $\mathcal{M}$ is **chromatic** if

$$\alpha_{\mathsf{upd}} \colon M \times C \to M.$$

Chaotic structures may be more succinct[1] but harder to reason with.
$\rightsquigarrow$ In what follows, memory structures are **chromatic**.

Given $\mathcal{M}$ and an arena $\mathcal{A} = (V_1, V_2, E)$, a *next-action function*

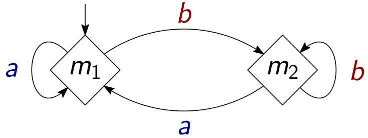
$$\alpha_{\mathsf{nxt}} \colon V_\ell \times M \to E$$

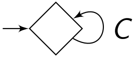
defines a strategy.

[1] Casares, "On the Minimisation of Transition-Based Rabin Automata and the Chromatic Memory Requirements of Muller Conditions", 2022.

Examples

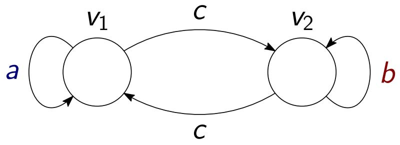
E.g., chromatic structure to remember whether a or b was last seen:

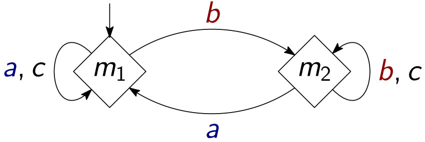


Memoryless strategies use **memory structure**  .

Example

$$C = \{a, b, c\},$$
$$W = \{w \in C^\omega \mid a \text{ is seen } \infty\text{ly often and } b \text{ is seen } \infty\text{ly often}\}$$





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but **two memory states** do! There is a winning strategy $\sigma\colon V_1 \times M \to E$.

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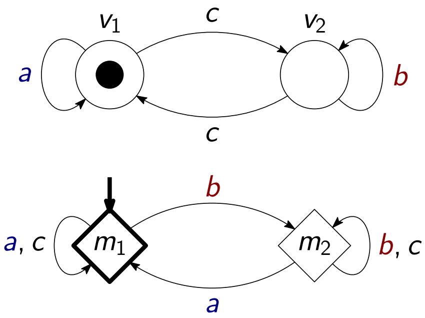

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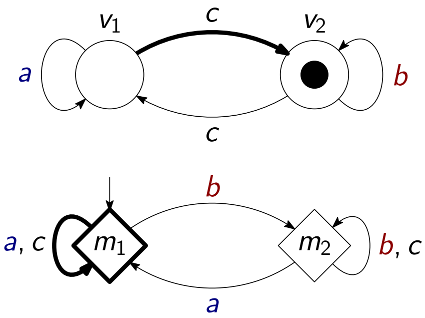

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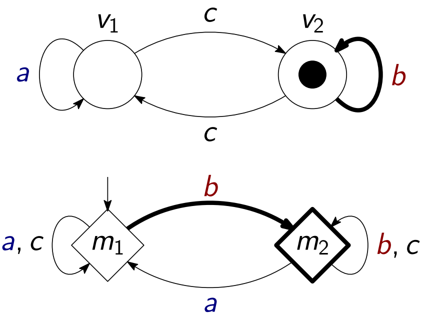

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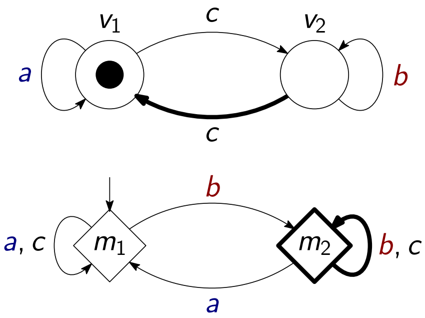

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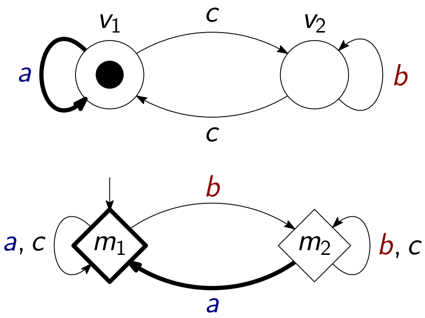

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Finite-memory determinacy

Memoryless determinacy

An **objective** is **memoryless-determined** if **in all arenas**, **memoryless** strategies suffice **for both players**.

Finite-memory determinacy

An **objective** is **finite-memory-determined** if **in all arenas**, **finite-memory** strategies suffice **for both players**.

We require *uniformity* of the strategies.

Various definitions depending on

- the class of **arenas** considered (finite, infinite, finitely branching...),
- whether we focus on **both** players or a **single** player.

State of the art: memoryless determinacy

Many “classical” objectives are **memoryless-determined**: reachability, Büchi, parity, energy, mean payoff, discounted sum...

Memoryless determinacy is well-understood:

- **Sufficient conditions** for **both** players,[2] for **a single** player.[3]
- **Characterizations** for **both** players over finite[4]/infinite[5] arenas, for **a single** player over infinite arenas.[6]

[2] Gimbert and Zielonka, “When Can You Play Positionally?”, 2004; Aminof and Rubin, “First-cycle games”, 2017.

[3] Kopczyński, “Half-Positional Determinacy of Infinite Games”, 2006; Bianco et al., “Exploring the boundary of half-positionality”, 2011.

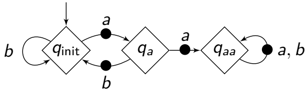
[4] Gimbert and Zielonka, “Games Where You Can Play Optimally Without Any Memory”, 2005.

[5] Colcombet and Niwiński, “On the positional determinacy of edge-labeled games”, 2006.

[6] Ohlmann, “Characterizing Positionality in Games of Infinite Duration over Infinite Graphs”, 2023.

State of the art: finite-memory determinacy

- **Finite-memory determinacy** is understood for specific objectives,[7] but few results of wide applicability.[8]
- Central class: **ω-regular objectives**. Examples with $C = \{a, b\}$:

ω-regular expressions	ω-automata	Linear temporal logic (LTL)
$b^* a b^* a C^\omega$		$\mathbf{GF}a$

Theorem[9, 10]

All ω-**regular objectives** are finite-memory-determined.

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[7] Chatterjee, Randour, and Raskin, "Strategy synthesis for multi-dimensional quantitative objectives", 2014; Colcombet, Fijalkow, and Horn, "Playing Safe", 2014; Bouyer, Hofman, et al., "Bounding Average-Energy Games", 2017.

[8] Le Roux, Pauly, and Randour, "Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions", 2018; Bouyer, Le Roux, and Thomasset, "Finite-Memory Strategies in Two-Player Infinite Games", 2022.

[9] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[10] Rabin, "Decidability of Second-Order Theories and Automata on Infinite Trees", 1969.

Significance

Consequences of a fine-grained understanding of **strategy complexity**:

- **Decidability** of logical theories through FM det. (see *monadic second-order logic*, linked to ω-regular objectives[11]).
- Practical **synthesis** problems through FM det. (see, e.g., *LTL* specifications[12]).
- At the core of algorithms to **solve** games (see, e.g., *parity games*[13]).
- Controllers as **compact** as possible.

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[11] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

[12] Pnueli, "The Temporal Logic of Programs", 1977.

[13] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

Overview of our contributions

I. General conditions for finite-memory determinacy

- **Arbitrary** objectives
- Algebraic **characterizations** of the sufficiency of a memory structure for both players
- **Theoretical tools** to help find memory structures
- Generalizations of memoryless determinacy results

II. Precise memory requirements of classes of objectives

- ω-**regular** objectives
- **Observation**: memory requirements not settled

1. *Regular* objectives ($\approx$ DFAs)
2. Objectives recognizable by *deterministic Büchi automata*

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- **Arbitrary** objectives
- Algebraic **characterizations** of the sufficiency of a memory structure for both players
- **Theoretical tools to help find memory structures**
  1. **One-to-two-player lifts**
  2. Memory structures $\rightsquigarrow$ automata for the objectives
- Generalizations of memoryless determinacy results

One-to-two-player lift

One-to-two-player memoryless lift (finite arenas)[14]

Let $W \subseteq C^\omega$ be an objective. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has memoryless optimal strategies,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has memoryless optimal strategies,

then both players have memoryless optimal strategies in **two-player** arenas.

Complexity does not increase if an opponent is added!
Easy to recover **memoryless determinacy** of, e.g., **parity**[15] and **mean-payoff**[16] objectives.

What about **finite-memory determinacy**?

[14] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[15] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[16] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

What about finite-memory determinacy?

- **Counterexample** to a one-to-two-player lift for FM determinacy ☹.
- In the counterexample, **the size of the memory depends** on the size of the one-player **arenas**. **Motivates the restriction to**. . .

Arena-independent finite memory

An objective is *arena-independent finite-memory determined* if

there exists a memory structure $\mathcal{M}$ such that for all arenas $\mathcal{A}$,

strategies based on $\mathcal{M}$ suffice to win in $\mathcal{A}$.

- Requires **chromatic** memory structures.
- Still holds for ω-regular objectives!
- **One-to-two-player lift works**!

One-to-two-player finite-memory lift

One-to-two-player finite-memory lift (finite arenas)

Let $W \subseteq C^\omega$ be an objective, $\mathcal{M}$ be a memory structure. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has optimal strategies based on $\mathcal{M}$,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has optimal strategies based on $\mathcal{M}$,

then both players have optimal strategies based on $\mathcal{M}$ in **two-player** arenas.

One-to-two-player lifts

When does strategy complexity in two-player zero-sum games
reduce to strategy complexity in **one-player** games?

Arenas \ Str. comp.	Memoryless	**FM "$\exists\mathcal{M}\forall\mathcal{A}$"**	Mildly growing
Finite	**[GZ05]**[17]	**[BLORV22]**	[Koz22][18]
Infinite	**[CN06]**[19]	**[BRV23]**	
Finite stochastic	[GZ09][20]	**[BORV21]**	

[17] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[18] Kozachinskiy, "One-To-Two-Player Lifting for Mildly Growing Memory", 2022.

[19] For prefix-independent objectives; Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[20] Gimbert and Zielonka, "Pure and Stationary Optimal Strategies in Perfect-Information Stochastic Games with Global Preferences", 2009.

I. General conditions for finite-memory determinacy

- **Arbitrary** objectives
- Algebraic **characterizations** of the sufficiency of a memory structure for both players
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Link with automaton representation

Let $W \subseteq C^\omega$ be an objective.

(Almost) Myhill-Nerode congruence

For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \iff yz \in W$.

I.e., x and y have the same winning continuations; as good as each other.

Properties

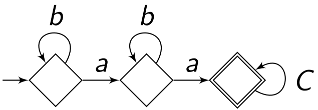
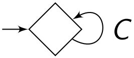
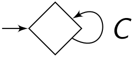
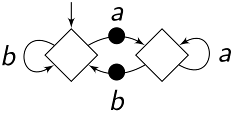
- If W is **ω-regular**, then $\sim_W$ has finitely many equivalence classes.
- There is a DFA $\mathcal{S}_W$ "**prefix classifier**" associated with $\sim_W$.

$\mathcal{S}_W$ might not "recognize" the language ($\neq$ languages of *finite* words)...

Two examples

... but we found a *decomposition* with **prefix classifier** × **memory structure**.

Let $C = \{a, b\}$.

Objective	**Prefix classifier $\mathcal{S}_W$**	**Sufficient memory**
$W = b^* a b^* a C^\omega$		
$W =$ "a and b ∞ly often"		

Main result

Let $W \subseteq C^\omega$ be an objective.

Theorem

If a finite memory structure $\mathcal{M}$ suffices to play optimally in **infinite** arenas for both players, then

$$W \textbf{ is recognized by a } \textit{parity automaton } (\mathcal{S}_W \otimes \mathcal{M}, p)$$

for some $p \colon M \times C \to \{0, \dots, n\}$.

In particular,

W is **chromatic-finite-memory-determined** over infinite arenas

$$\Longleftrightarrow$$

W is ω-**regular**.

Generalizes [CN06][21] (prefix-independent, memoryless case).

[21] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Part I: Summary

- Useful **notion** of *arena-independent* FM determinacy.
- General **characterizations** over finite and infinite arenas, theoretical **tools** to determine memory requirements.
- Central place of **ω-regular objectives**.

Related publications

- Bouyer, Le Roux, Oualhadj, Randour, V. (CONCUR'20 & LMCS) "Games Where You Can Play Optimally with Arena-Independent Finite Memory"
- Bouyer, Randour, V. (STACS'22 & TheoretiCS) "Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs"

Limits

Wide applicability, but...

- not fully **effective**;
- in general, no tight memory requirements for **each** player.

II. Precise memory requirements of classes of objectives

II. Precise memory requirements of classes of objectives

- ω-**regular** objectives
- **Observation**: memory requirements not settled

1. ***Regular* objectives ($\approx$ DFAs)**
   - **Effective characterization of precise memory structures**
   - **Existence of small structures is NP-complete**
2. Objectives recognizable by *deterministic Büchi automata*
   - Effective characterization of "no memory for $\mathcal{P}_1$"
   - Decidable in polynomial time

Regular objectives

Well-understood ω-regular objectives: *Muller conditions*, focusing on what is seen **infinitely often**.[22,23,24]
E.g., $b^*ab^*aC^\omega$ is not a Muller condition.

Missing pieces

Alternative quest: objectives where "**finite prefixes matter**".

We consider the "simplest" ones.

Regular objectives

- A **regular reachability objective** is a set LC^ω with $L \subseteq C^*$ regular.
- A **regular safety objective** is a set $C^\omega \setminus LC^\omega$.

Expressible as standard **deterministic finite automata**.

[22] Gurevich and Harrington, "Trees, Automata, and Games", 1982.

[23] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

[24] Casares, Colcombet, and Lehtinen, "On the Size of Good-For-Games Rabin Automata and Its Link with the Memory in Muller Games", 2022.

Question

Memory requirements of regular objectives

Characterize the memory structures that suffice to make optimal decisions for regular objectives **in any arena**. Compute **minimal** ones.

Ideas

- A DFA recognizing the language, taken as a memory structure, always suffices for both players
  (≈ usual approach: taking the product of the arena and the DFA).
- But can be much smaller in general!
- Properties linked to the Myhill-Nerode congruence.

Comparing words

Let $W \subseteq C^\omega$ be an objective.

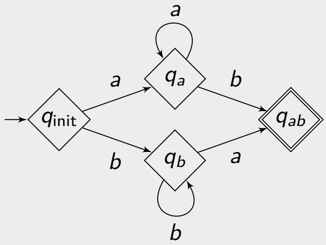
Comparing prefixes

For $x, y \in C^*$, $x \preceq_W y$ if for all $z \in C^\omega$, $xz \in W \implies yz \in W$.

I.e., y has more winning continuations than x; better situation.

Example

Let W be the regular **reachability** objective induced by this DFA.



E.g., $\varepsilon \prec_W a$,
a and b are *incomparable* for $\preceq_W$.

Necessary condition

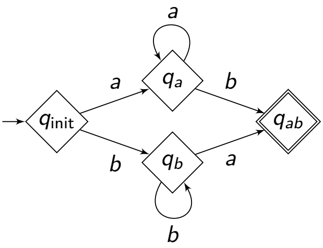
Let $W \subseteq C^\omega$ be an objective, $\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ be a memory structure.

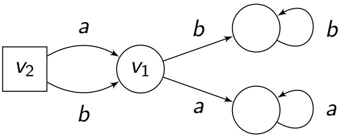
Lemma

For $\mathcal{M}$ to suffice for $\mathcal{P}_1$, W needs to be **$\mathcal{M}$-strongly-monotone** ("$\mathcal{M}$ distinguishes incomparable words"), i.e.,

if $x, y \in C^*$ are incomparable for $\preceq_W$,
then $\alpha^*_{\text{upd}}(m_{\text{init}}, x) \neq \alpha^*_{\text{upd}}(m_{\text{init}}, y)$.

Why? We can build an arena in which distinguishing x and y is critical.





Characterizations

Theorem

Let W be a **regular safety objective**.

A memory structure $\mathcal{M}$ suffices in all arenas for $\mathcal{P}_1$
if and only if
W is $\mathcal{M}$-strongly-monotone.

Theorem

Let W be a **regular reachability objective**.

Memory structure $\mathcal{M}$ suffices in all arenas for $\mathcal{P}_1$
if and only if
W is $\mathcal{M}$-strongly-monotone **and**
W is $\mathcal{M}$-*progress-consistent*.

Computational complexity

Decision problems

Input: An automaton $\mathcal{D}$ inducing the regular **reachability** (or **safety**) objective W and $k \in \mathbb{N}$.
Question: $\exists$ a memory structure $\mathcal{M}$ with $\leq k$ states that suffices for W?

Thanks to the "effectiveness" of the two properties, we showed that:

Theorem

These problems are NP-complete.

Implementation of algorithms[25] that find minimal memory structures for regular objectives, using a **SAT solver**.

[25] https://github.com/pvdhove/regularMemoryRequirements

II. Precise memory requirements of classes of objectives

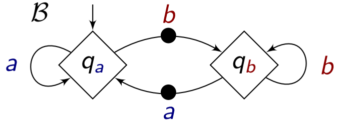
- ω-**regular** objectives
- **Observation**: memory requirements not settled

1. *Regular* objectives ($\approx$ DFAs)
   - Effective characterization of precise memory structures
   - Existence of small structures is NP-complete
2. **Objectives recognizable by *deterministic Büchi automata***
   - **Effective characterization of "no memory for $\mathcal{P}_1$"**
   - **Decidable in polynomial time**

Deterministic Büchi automata

A **deterministic Büchi automaton** $\mathcal{B}$ on C

- reads **infinite** words (in C^ω),
- accepts words that see infinitely many **Büchi transitions** •.



$$\mathcal{L}(\mathcal{B}) = \{w \in \{a, b\}^\omega \mid w \text{ sees } \infty\text{ly many } a \text{ and } \infty\text{ly many } b\}$$

Question

Given $\mathcal{B}$, **can $\mathcal{P}_1$ win without memory for objective $W = \mathcal{L}(\mathcal{B})$**?
(Is $\mathcal{L}(\mathcal{B})$ *half-positional*?)

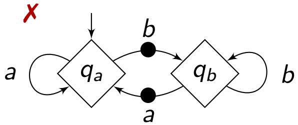
Results

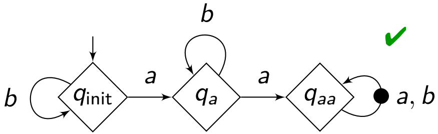
Let $\mathcal{B}$ be a **deterministic Büchi automaton**.

Theorem

For objective $W = \mathcal{L}(\mathcal{B})$, $\mathcal{P}_1$ does not need memory **if and only if**

- all prefixes are comparable for $\preceq_W$,
- W *is progress-consistent*, and
- W ***is recognized by its prefix classifier as a DBA***.





Polynomial-time algorithm

Can be **decided** in $\mathcal{O}(|\mathcal{B}|^4)$ time.

Part II: Summary

- Tools to study memory requirements of classes of ω-regular objectives.
- Effective **characterizations** for DFAs and DBAs.
- **Decidability** and **complexity** of the related decision problems.

Related publications

- Bouyer, Fijalkow, Randour, V. (Accepted to ICALP'23) "How to Play Optimally for Regular Objectives?"
- Bouyer, Casares, Randour, V. (CONCUR'22, invited to LMCS) "Half-Positional Objectives Recognized by Deterministic Büchi Automata"

Future works

- (*Part I*) General results for arena-*dependent* memory requirements.
  - Observing *edges* rather than colors.
  - Well-behaved nondeterminism (*history-determinism*).[26]
- (*Part II*) Automatically **compute minimal memory structures** for *all* ω-regular objectives?
- More expressive **settings** (e.g., concurrent[27] games).
- More expressive **strategy models** (e.g., pushdown[28] automata).

Thanks!

[26] Boker and Lehtinen, "When a Little Nondeterminism Goes a Long Way: An Introduction to History-Determinism", 2023.

[27] Bordais, Bouyer, and Le Roux, "Optimal Strategies in Concurrent Reachability Games", 2022.

[28] Walukiewicz, "Pushdown Processes: Games and Model-Checking", 2001.