Memory Requirements of Omega-Regular Objectives: the Regular Case

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Outline

Synthesis problem

Synthesizing **controllers** for **reactive systems** with an **objective**.
Systems and their environment modeled with **zero-sum games**.

Strategy complexity

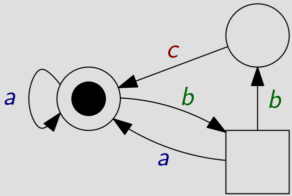
Given an objective, what are the **smallest** optimal controllers?
$\rightsquigarrow$ What is the *smallest automatic structure* that remembers **sufficient information** to make **optimal decisions**?

Results

Characterization of automatic structures for *regular objectives*;
computational complexity of finding small structures.

Games

Zero-sum turn-based games on graphs



- $C = \{a, b, c\}$, $\mathcal{A} = (V_1, V_2, E)$.
- Two players $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$)

Strategies

A **strategy** of $\mathcal{P}_1$ is a function $\sigma \colon E^* \to E$.
A strategy σ is **winning for W from $v \in V$** if all infinite paths from v *consistent with* σ induce an infinite word in W.

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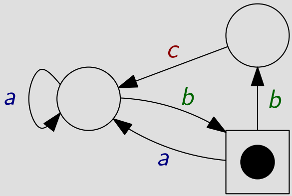

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  $w = b$

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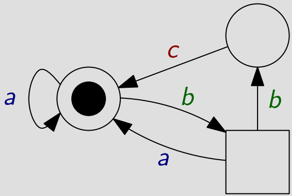

- $C = \{a, b, c\}$, $\mathcal{A} = (V_1, V_2, E)$.
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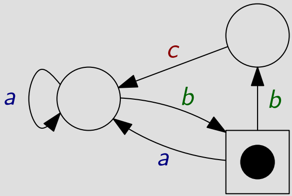

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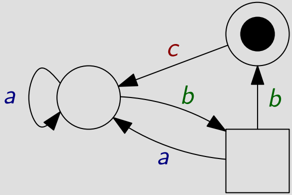

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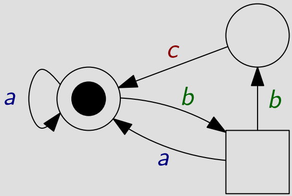

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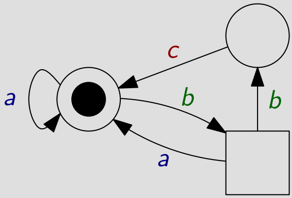

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- Two players $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$) generate an infinite word $w = babbc\ldots \in C^\omega$.
- **Objective** of $\mathcal{P}_1$ is a set $W \subseteq C^\omega$.
- **Zero-sum**: objective of $\mathcal{P}_2$ is $C^\omega \setminus W$.

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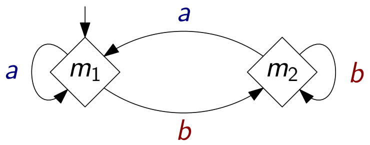
Representations of a strategy

In general, a strategy $\sigma\colon E^* \to E$ has an **infinite** representation.
For synthesis, we like when winning strategies admit a **finite** representation with a **computable** size. Usual finite representation:

Memory structure

Memory structure $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$: finite set of states M, initial state m_{init}, update function $\alpha_{\mathsf{upd}}\colon M \times C \to M$.

Ex.: remember whether a or b was last played (**not yet a strategy!**):

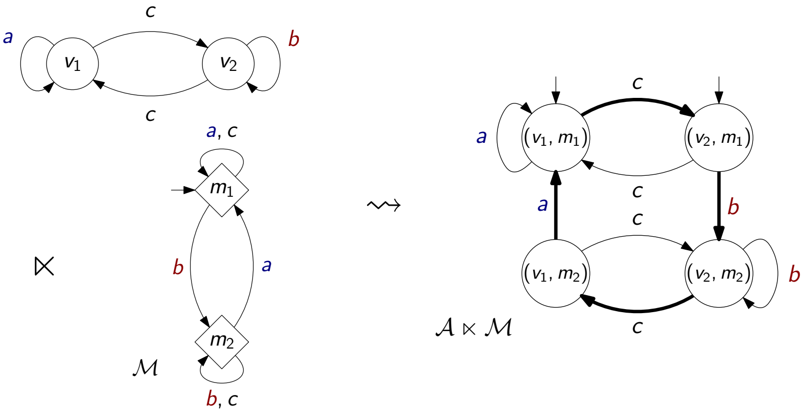


Given an arena $\mathcal{A} = (V_1, V_2, E)$: *next-action function* $\alpha_{\mathsf{nxt}}\colon V_i \times M \to E$.

Finite memory ≈ no memory in the product

Memory $\mathcal{M}$ in $\mathcal{A}$ ≈ no memory in arena $\mathcal{A} \ltimes \mathcal{M}$.

If $C = \{a, b, c\}$,
$W = \{w \in C^\omega \mid a$ is seen ∞ly often and b is seen ∞ly often$\}$:



ω-regular objectives

The *ω-regular objectives* are the ones that can be expressed with **ω-regular expressions**, or equivalently, the ones that can be expressed by **deterministic Muller automata**.

Examples with $C = \{a, b\}$:

- $W = b^* a b^* a C^\omega$;
- $W = (b^* a)^\omega$;
- $W = ((ab) \mid (ba)) C^\omega$.

Theorem (Büchi, Landweber, 1969)[1]

All **ω-regular objectives** admit **finite-memory winning strategies** in all arenas.

[1] Büchi and Landweber, "Definability in the Monadic Second-Order Theory of Successor", 1969.

Well-studied case: *Muller conditions*

For $\mathcal{F} \subseteq 2^C$, *objective* Muller($\mathcal{F}$) is the set of words whose set of colors seen infinitely often is in $\mathcal{F}$.

Examples with $C = \{a, b\}$:

- $\text{Muller}(\{\{a\}, \{a, b\}\}) = (b^* a)^\omega$,
- $\text{Muller}(\{\{a, b\}\}) = (b^* a)^\omega \cap (a^* b)^\omega$.

Memory requirements of Muller conditions

- First upper bound of size $\mathcal{O}(|C|!)$ in 1982 (*later appearance record*);[2]
- Followed by many works about specific cases;[3,4]
- **Characterization** of precise memory requirements and **algorithm** to compute them in 1997 ([DJW97][5]).

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[2] Gurevich and Harrington, "Trees, Automata, and Games", 1982.

[3] Emerson and Jutla, "Tree Automata, Mu-Calculus and Determinacy (Extended Abstract)", 1991.

[4] Klarlund, "Progress Measures, Immediate Determinacy, and a Subset Construction for Tree Automata", 1994.

[5] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

Is that it?

We have:

1. that all ω-regular objectives can be represented by a **deterministic automaton using a Muller acceptance condition**;
2. a complete understanding of the **memory requirements of Muller conditions**.

Does this settle the question of the memory requirements of all ω-regular objectives?

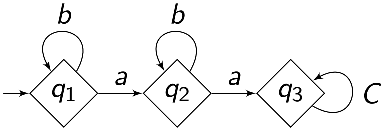
Sometimes quoted as such,[6] but **not the case** (it is only an upper bound)!

[6] In *Handbook of Model Checking* (Bloem, Chatterjee, and Jobstmann, "Graph Games and Reactive Synthesis", 2018): "The results of Dziembowski et al. [80] give precise memory requirements for strategies in 2-player games with ω-regular objectives".

Why only an upper bound?

Let $C = \{a, b\}$, $W = b^* a b^* a C^\omega$ ($\approx$ *seeing a two or more times*).
Express it as a **Muller condition**?

Not directly a Muller condition Muller($\mathcal{F}$) with $\mathcal{F} \subseteq 2^C$
$\rightsquigarrow$ needs an **automaton structure**.



$\rightsquigarrow W = \text{Muller}(\{F \subseteq \{q_1, q_2, q_3\} \mid q_3 \in F\})$.

Using [DJW97],[7] we need 1 memory state...

... **after** augmenting the game graph with the automaton,
so **upper bound of 3 states of memory**.

But 1 **memory state** suffices for winning strategies!

[7] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

Orthogonal quest: **regular** objectives

How to go further?

Study the memory requirements of ω-regular objectives with **non-trivial automaton structures**.

We consider "the simplest ones".

Regular objectives

- A **regular reachability objective** is a set LC^ω with $L \subseteq C^*$ regular.
- A **regular safety objective** is a set $C^\omega \setminus LC^\omega$.

- A player wants to realize a word in L, the other wants to prevent it.
- Expressible as standard **deterministic finite automata**.
- *Special cases of open and closed sets, at the first level of the Borel hierarchy*.

Comparing words

Let $W \subseteq C^\omega$ be an objective.

Winning continuations

For $x \in C^*$, $x^{-1}W = \{w \in C^\omega \mid xw \in W\}$.

For $x, y \in C^*$,

- $x \sim_W y$ if $x^{-1}W = y^{-1}W$ ($\approx$ Myhill-Nerode equivalence relation),
- $x \preceq_W y$ if $x^{-1}W \subseteq y^{-1}W$ (called **prefix preorder**).

Blackboard example for a regular safety objective.

Necessary condition for the memory

Let W be an objective.

Lemma

The memory structure $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ needs to **distinguish incomparable words** (for $\preceq_W$), i.e.,

$$\text{if } x, y \in C^* \text{ are incomparable for } \preceq_W, \\ \text{then } \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, x) \neq \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, y).$$

Why?

Blackboard example.

In other words, the memory structure needs to be able to know a chain for $\preceq_W$ in which we are, but it is OK not to remember the precise automaton state.

Characterization: safety

Let W be a **regular** **safety** **objective**.

Theorem

A memory structure suffices to win in all arenas for W **if and only if** it distinguishes incomparable words.

Question

How to find the smallest such memory structure?

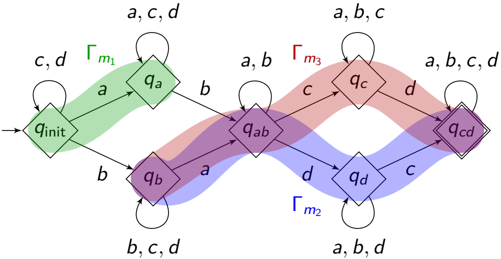
A more involved example

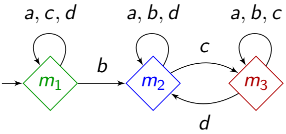
Blackboard example.

Two constructions that always work for upper bounds:

1. just take the whole automaton as a memory structure;
2. build a memory state for each maximal chain.

Possible to do better? **Yes**!





Reformulation using **chain coverings**

Let W be the regular safety objective derived from automaton $\mathcal{D} = (Q, C, q_{\mathsf{init}}, \delta, F)$.

Lemma

There is a memory structure $\mathcal{M}$ with k states that suffices for W *if and only if* there are k sets $\Gamma_1, \ldots, \Gamma_k \subseteq Q$ such that

1. $Q = \bigcup_{i=1}^{k} \Gamma_i$,
2. for all $i \in \{1, \ldots, k\}$, Γ_i is a chain for $\preceq_W$, and
3. for all $i \in \{1, \ldots, k\}$, for all $c \in C$, there is $j \in \{1, \ldots, k\}$ such that $\delta(\Gamma_i, c) \subseteq \Gamma_j$.

Computational complexity: safety

Decision problem

MEMORYSAFE
Input: An automaton $\mathcal{D}$ inducing the regular safety objective W and an integer $k \in \mathbb{N}$.
Question: Does there exist a memory structure $\mathcal{M}$ with at most k states which suffices to play optimally for W?

Thanks to the reformulation, we get that MEMORYSAFE is in NP. We also showed NP-hardness thanks to a reduction from HAMILTONIANCYCLE.

Theorem

MEMORYSAFE is NP-complete.

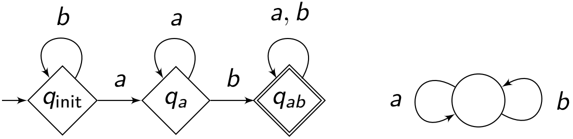
Regular reachability

Let W be a regular **reachability** objective.

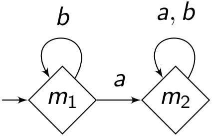
Memory structures still need to distinguish incomparable words.

But not sufficient!

$W = b^* a a^* b C^\omega$:



Main idea: seeing a is necessary and *makes progress*. However, we cannot just play a to win. Word a is an *insufficient progress*.
This memory structure distinguishes this insufficient progress.



Condition necessary for reachability

Let $W \subseteq C^\omega$ be an objective.

Necessary property

Let $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ be a memory structure.
Memory structure $\mathcal{M}$ **distinguishes insufficient progress** if for all $w_1 \in C^*$ with $m = \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, w_1)$, for all $w_2 \in C^*$, if $w_1(w_2)^\omega \notin W$ and $w_1 \prec w_1 w_2$, then $\alpha^*_{\mathsf{upd}}(m, w_2) \neq m$.

Necessary for $\mathcal{M}$ to be optimal. Why?

Blackboard proof.

Characterization: reachability

Let W be a **regular reachability objective**.

Theorem

Memory structure $\mathcal{M}$ suffices to win in all arenas if and only if $\mathcal{M}$ *distinguishes incomparable words* and $\mathcal{M}$ *distinguishes insufficient progress*.

Remark

Distinguishing insufficient progress is necessary for all objectives, even for regular safety ones...
... but there is *no insufficient progress* for regular safety objectives!

Computational complexity: reachability

Decision problem

MemoryReach
Input: An automaton $\mathcal{D}$ inducing the regular reachability objective W and an integer $k \in \mathbb{N}$.
Question: Does there exist a memory structure $\mathcal{M}$ with at most k states which suffices to play optimally for W?

Theorem

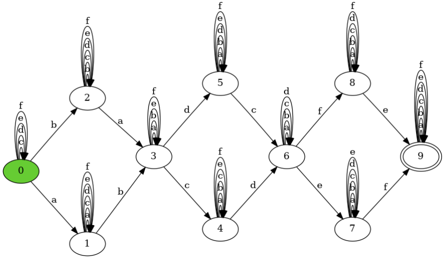
MemoryReach is NP-complete.

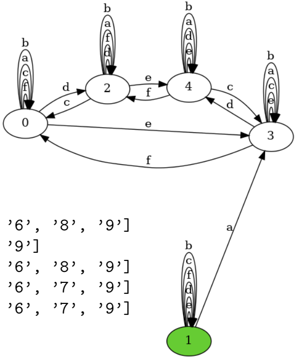
Needed to show that "$\mathcal{M}$ distinguishes insufficient progress" is in NP, but the same hardness proof as for MemorySafe.

Implementation

We have implemented algorithms that automatically find minimal memory structures for regular objectives. **Simple ideas**: binary search on the minimal size, encoding properties as SAT instances and use of a SAT solver.

```
D =
```






```
M = memReq.smallest_memory_safety(D)
```

```
\Gamma_0 = [’1’, ’3’, ’4’, ’6’, ’8’, ’9’]
\Gamma_1 = [’0’, ’2’, ’7’, ’9’]
\Gamma_2 = [’1’, ’3’, ’5’, ’6’, ’8’, ’9’]
\Gamma_3 = [’1’, ’3’, ’4’, ’6’, ’7’, ’9’]
\Gamma_4 = [’1’, ’3’, ’5’, ’6’, ’7’, ’9’]
```

Conclusion

Summary

- Characterization of the memory structures for **regular objectives**.
- **NP-completeness** of finding small memory structures.
- Implementation using a SAT solver.

Future work

- Two orthogonal directions are understood: *Muller conditions*[8] and *regular objectives*.[9]
  $\rightsquigarrow$ What about objectives recognized by deterministic Muller automata, i.e., **ω-regular objectives**?
- Partial recent results for objectives recognized by **deterministic Büchi automata** and **memoryless strategies**.[10]

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[8] Dziembowski, Jurdziński, and Walukiewicz, "How Much Memory is Needed to Win Infinite Games?", 1997.

[9] Bouyer, Fijalkow, et al., "How to Play Optimally for Regular Objectives?", 2022.

[10] Bouyer, Casares, et al., "Half-Positional Objectives Recognized by Deterministic Büchi Automata", 2022.