Characterizing ω-regular languages through strategy complexity of games on infinite graphs **[Ongoing work]**

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Outline

Strategy synthesis for zero-sum turn-based games on graphs

Design **optimal** controllers for systems interacting with an **antagonistic** environment.

"Optimal" w.r.t. an objective or a specification.

Interest in "simple" controllers

Finite-memory determinacy: when do **finite-memory** strategies suffice?

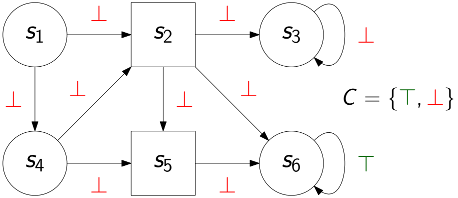
Inspiration

Results about **memoryless** **determinacy**.[1,2]

[1] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[2] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Zero-sum turn-based games on graphs



- Two-player **arenas**: S_1 ($\bigcirc$, for $\mathcal{P}_1$) and S_2 ($\square$, for $\mathcal{P}_2$), edges E.
- Set C of **colors**. Edges are colored.
- **Objectives** given by a set $W \subseteq C^\omega$. **Zero-sum**.
- A strategy for $\mathcal{P}_i$ is a (partial) function $\sigma\colon E^* \to E$.

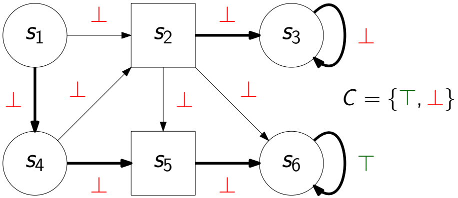
First: **finite** arenas.

Memoryless determinacy

Question

Given an objective, do “simple” strategies suffice to play optimally in all arenas?

A strategy σ of $\mathcal{P}_i$ is *memoryless* if it is a function ~~E^*~~ $S_i \to E$.



E.g., for reachability, **memoryless** strategies suffice to play optimally.

Also suffice for safety, Büchi, co-Büchi, parity, mean payoff, energy. . .

Memoryless determinacy

Memoryless determinacy

An objective $W \subseteq C^\omega$ is **memoryless-determined** if memoryless strategies suffice to play optimally for both players in all (finite) arenas.

We require *uniformity*: a **single** memoryless strategy must be winning from all the states where that is possible.

Memoryless determinacy

Good understanding of **memoryless determinacy**:

- **sufficient** conditions to guarantee memoryless optimal strategies for **both** players.[3,4]
- **sufficient** conditions to guarantee memoryless optimal strategies for **one** player.[5,6,7,8]
- **characterization of the objectives admitting optimal memoryless strategies for both players.**[9]

[3] Gimbert and Zielonka, "When Can You Play Positionally?", 2004.

[4] Aminof and Rubin, "First-cycle games", 2017.

[5] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006.

[6] Gimbert, "Pure Stationary Optimal Strategies in Markov Decision Processes", 2007.

[7] Bianco et al., "Exploring the boundary of half-positionality", 2011.

[8] Gimbert and Kelmendi, "Submixing and Shift-Invariant Stochastic Games", 2014.

[9] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

Gimbert and Zielonka's characterization

One-to-two-player memoryless lift (finite arenas)[10]

Let $W \subseteq C^\omega$ be an objective. If

- in all **one-player** arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has an optimal memoryless strategy,
- in all **one-player** arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has an optimal memoryless strategy,

then both players have an optimal memoryless strategy in all **two-player** arenas.

Extremely useful in practice. Very easy to recover memoryless determinacy of, e.g., parity and **mean-payoff** games.

[10] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

Application: memoryless determinacy of **mean payoff**[12]

- Colors $C = \mathbb{Q}$. Objective $W \subseteq C^\omega$ (for $\mathcal{P}_1$):
  obtain a **mean payoff** (average weight by transition) ≥ 0.

- In **one-player** arenas, simply **reach** and loop around the **simple cycle** with the **greatest** (for $\mathcal{P}_1$) or **smallest** (for $\mathcal{P}_2$) **mean payoff**
  $\rightsquigarrow$ memoryless strategy.
  Memoryless strategies can be uniformized as W is prefix-independent.[11]

$\xrightarrow{[GZ05]}$ Memoryless strategies also suffice to play optimally in **two-player** arenas!

[11] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[12] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

What about **infinite arenas**?

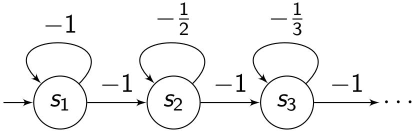
Motivations

- Links between the **strategy complexity** in finite and **infinite** arenas?
- Can we get a similar **one-to-two-player lift** for **infinite** arenas?
  $\rightsquigarrow$ proof technique for finite arenas (induction on number of edges) is not suited to infinite arenas.

Greater memory requirements in **infinite** arenas

Objective W: get a mean payoff ≥ 0.

- **Memoryless** strategies suffice in **finite** arenas.
- **Infinite** memory is required in (even one-player) deterministic **infinite** arenas.[13]



$\rightsquigarrow$ Possible to get 0 at the limit **with infinite memory**: loop $\left\lceil e^{e^n} \right\rceil$ many times in state s_n for all n.

[13] Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, 1994.

Infinite arenas, memoryless strategies

Let $W \subseteq C^\omega$ be a **prefix-independent** objective.

Characterization of memoryless determinacy (infinite arenas)[14]

If **memoryless strategies** suffice to play optimally for both players in **infinite arenas**, then W is a **parity condition**.

Parity condition: there exists $p\colon C \to \{0, \ldots, n\}$ such that

$$w = c_1 c_2 \ldots \in W \iff \limsup_i p(c_i) \text{ is even.}$$

Characterization since parity objectives are memoryless-determined (in arenas of any cardinality).[15]

[14] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[15] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

First insight

Possible to obtain the result with a hypothesis on **one-player** arenas only!
Let $W \subseteq C^\omega$ be a **prefix-independent** objective.

Characterization of memoryless determinancy (infinite arenas)

If **memoryless strategies** suffice to play optimally for both players in **one-player infinite arenas**, then W is a **parity condition**.

Proof of **one-to-two-player lift**:

Memoryless determinacy in one-player infinite arenas
$\implies$ W is a parity condition
$\implies$ memoryless determinacy (in two-player infinite arenas).

Two "limits" of the result

- There are simple memoryless-determined objectives (in infinite arenas) which are not **prefix-independent** (e.g., reachability).
  A bit disappointing to miss memoryless-determined objectives.

- What about **strategies** with **finite memory**?
  $\rightsquigarrow$ more and more prevalent in the literature.

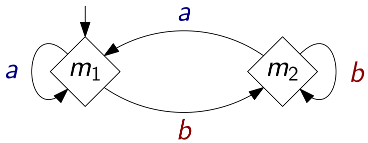
Finite memory

Finite-memory strategy $\approx$ memory structure + next-action function.

Memory structure

Memory structure $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$: finite set of states M, initial state m_{init}, update function $\alpha_{\mathsf{upd}}\colon M \times C \to M$.

Ex. to remember whether a or b was last played (**not yet a strategy!**):



Given an arena $\mathcal{A} = (S, S_1, S_2, E)$: *next-action function* $\alpha_{\mathsf{nxt}}\colon S_i \times M \to E$.
Memoryless strategies are based on the **"trivial" memory structure**.

Finite-memory determinacy

Finite-memory determinacy

An objective W is **finite-memory-determined** if **there exists** **a finite memory structure** $\mathcal{M}$ such that strategies based on memory $\mathcal{M}$ suffice to play optimally for both players **for all** **arenas** $\mathcal{A}$.

Remark

Usually, the definition inverts the order of the quantifiers. The order has a big impact in **finite arenas**,[16] but not in **infinite arenas**.

[16] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2020.

One-to-two-player lifts

When does two-player zero-sum memory determinacy reduce to **one-player** memory determinacy?

Arenas\Str. comp.	Memoryless	**FM "$\exists\mathcal{M}\forall\mathcal{A}$"**	Mildly growing
Finite deterministic	**[GZ05]**[17]	[BLORV20][18]	[Koz21][19]
Finite stochastic	[GZ09][20]	[BORV21][21]	
Infinite determin.	**P-Ind: [CN06]**[22]	**New work**	

[17] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

[18] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2020.

[19] Kozachinskiy, "One-to-Two-Player Lifting for Mildly Growing Memory", 2021.

[20] Gimbert and Zielonka, "Pure and Stationary Optimal Strategies in Perfect-Information Stochastic Games with Global Preferences", 2009.

[21] Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2021.

[22] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Tool to get rid of prefix-independence: right congruence

Let L be a language of **finite** words on alphabet C.

Right congruence

For $x, y \in C^*$, $x \sim_L y$ if for all $z \in C^*$, $xz \in L \Leftrightarrow yz \in L$.

Myhill-Nerode theorem[23]

L is **regular** if and only if $\sim_L$ has **finite index**.
The equivalence classes of $\sim_L$ correspond to the states of the minimal DFA for L.

[23] Nerode, "Linear Automaton Transformations", 1958.

Tool to get rid of prefix-independence: right congruence

Let W be a language of **infinite** words (= an objective) on alphabet C.

Right congruence

For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \Leftrightarrow yz \in W$.

Links with ω-regularity?

- If W is **ω-regular**, then $\sim_W$ has finite index.
  In this case, there is still a DFA $\mathcal{M}_\sim$ associated with $\sim_W$.
- The reciprocal is not true.

Examples:

- $C = \mathbb{Q}$, $W = \mathsf{MP}^{\geq 0}$: $\sim_{\mathsf{MP}^{\geq 0}}$ has index 1 but not ω-regular;
- $C = \{a, b\}$, $W = C^*(ab)^\omega$ is ω-regular but $\mathcal{M}_\sim$ is not "useful";
- $C = \{a, b\}$, $W = C^*aC^*aC^\omega$: ω-regular and $\mathcal{M}_\sim$ is "useful".

Memory requirements in **infinite** arenas

- $C = \mathbb{Q}$, $W = \mathsf{MP}^{\geq 0}$: $\sim_{\mathsf{MP}^{\geq 0}}$ has index 1 but not ω-regular: **infinite memory**;
- $C = \{a, b\}$, $W = C^*(ab)^\omega$ is ω-regular but $\mathcal{M}_\sim$ is not "useful": the minimal memory (blackboard) is useful;
- $C = \{a, b\}$, $W = C^*aC^*aC^\omega$: ω-regular and $\mathcal{M}_\sim$ is "useful": memoryless-determined.

Insight for prefix-independence

Let W be an objective.

Replacement for prefix-independence

If a finite memory structure suffices to play optimally in **one-player** infinite arenas for both players, then $\sim_W$ **has finite index** (so $\mathcal{M}_\sim$ is finite).

Intuition: even without assuming prefix-independence (index of $\sim_W$ is 1), we have a strong property on prefixes for free (index of $\sim_W$ is finite).

Main result

Let W be an objective.

Theorem

If a finite memory structure $\mathcal{M}$ suffices to play optimally in **one-player** infinite arenas for both players, then W is recognized by a **parity automaton** $(\mathcal{M}_\sim \otimes \mathcal{M}, p)$.

$\rightsquigarrow$ if $\mathcal{M}_\sim \otimes \mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$,

$$p\colon M \times C \to \{0, \dots, n\}.$$

Generalizes [CN06][24] ($\mathcal{M}_\sim = \mathcal{M} = \mathcal{M}_{\mathsf{triv}}$).

[24] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

Corollaries

Let $W \subseteq C^\omega$ be an objective.

One-to-two-player FM lift (infinite arenas)

If W is finite-memory-determined in **one-player** infinite arenas, then W is finite-memory-determined in **two-player** infinite arenas.

Characterization

W is **finite-memory-determined** if and only if W is ω-**regular**.

Proof: W is finite-memory-determined in **one-player** arenas

$\Longrightarrow$ W is recognized by a deterministic parity automaton (ω-regular).

$\Longrightarrow$ this parity automaton (as a memory) suffices in **two-player** arenas.[25]

$\Longrightarrow$ this parity automaton (as a memory) suffices in **one-player** arenas.

[25] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

Summary

Contributions

- New **one-to-two-player lift** for zero-sum games on infinite graphs.
- **Strategic characterization** of ω-regular languages.

Future work

- Other classes of arenas (e.g., finitely branching).
- Stochastic infinite arenas?

Thanks! Questions?