# Characterizing $\omega$-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs

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# Outline

### Strategy synthesis for zero-sum turn-based games

Design **optimal** controllers for systems interacting with an **antagonistic** environment.

### Interest in "simple" strategies

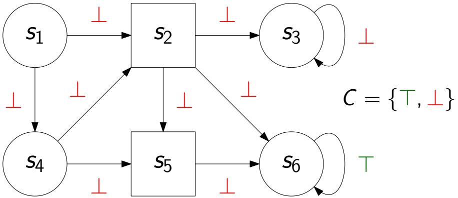
**Finite-memory determinacy**: when do **finite-memory** strategies suffice?
Focus on games on **infinite** graphs.

### Inspiration

Results about **memoryless** **determinacy**.[1]

[1] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

# Zero-sum turn-based games on graphs



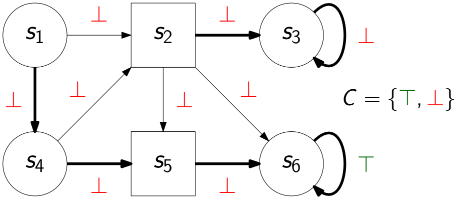

- Two-player **arenas**: $S_1$ ($\bigcirc$, for $\mathcal{P}_1$) and $S_2$ ($\square$, for $\mathcal{P}_2$), edges $E$.
- Set $C$ of **colors**. **Edges** are colored.
- **Objectives** are sets $W \subseteq C^\omega$. **Zero-sum**.
- **Strategy** for $\mathcal{P}_i$: function $\sigma\colon E^* \to E$.

# Memoryless determinacy

## Question

For an objective, do *simple* strategies suffice to play optimally in all arenas?

A strategy $\sigma$ of $\mathcal{P}_i$ is *memoryless* if it is a function $\cancel{E^*}\, S_i \to E$.




E.g., for Reach($\top$), **memoryless** strategies suffice to play optimally.

Also suffice for Büchi, parity... objectives.

# Memoryless determinacy

**Good understanding of memoryless determinacy in finite arenas**

**Sufficient** conditions and **characterizations** of memoryless determinacy

- for **one** player,[2,3,4,5]
- for **both** players.[6,7,8]

What about **infinite** arenas?

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[2] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006.

[3] Gimbert, "Pure Stationary Optimal Strategies in Markov Decision Processes", 2007.

[4] Bianco et al., "Exploring the boundary of half-positionality", 2011.

[5] Gimbert and Kelmendi, "Submixing and Shift-Invariant Stochastic Games", 2014.

[6] Gimbert and Zielonka, "When Can You Play Positionally?", 2004.

[7] Aminof and Rubin, "First-cycle games", 2017.

[8] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

# What about **infinite arenas**?

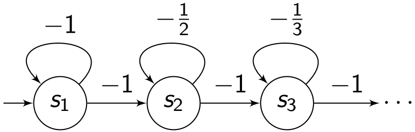
## Motivations

1. Links between the **strategy complexity** in finite and **infinite** arenas?
2. Similar sufficient conditions/characterizations for **infinite** arenas?
   $\rightsquigarrow$ Classical proof technique for finite arenas (induction on number of edges) not suited to infinite arenas.

# Greater memory requirements in **infinite** arenas

Colors $C = \mathbb{Q}$, objective $W =$ "get a *mean payoff* $\geq 0$".

- **Memoryless** strategies sufficient in **finite** arenas.[9]
- **Infinite** memory required in **infinite** arenas.[10]




$\rightsquigarrow$ Possible to get 0 at the limit **with infinite memory**:
loop increasingly many times in states $s_n$.

[9] Ehrenfeucht and Mycielski, "Positional Strategies for Mean Payoff Games", 1979.

[10] Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, 1994.

# Nice result

Let $W \subseteq C^\omega$ be a **prefix-independent** objective.

**Characterization of memoryless determinacy**[11]

If **memoryless strategies** suffice to play optimally for **both** players in all **infinite arenas**, then $W$ is a **parity condition**.

**Parity condition**: there exists $p\colon C \to \{0, \ldots, n\}$ such that

$$w = c_1 c_2 \ldots \in W \iff \limsup_i p(c_i) \text{ is even.}$$

**Characterization** since parity conditions are memoryless-determined.[12]

[11] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

[12] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

# Two possible extensions

1. What about **strategies** with **finite memory**?
   $\rightsquigarrow$ More and more prevalent in the literature.

2. Some simple memoryless-determined objectives are not **prefix-independent** (e.g., Reach($\top$)).
   $\rightsquigarrow$ This characterization **misses** memoryless-determined objectives.

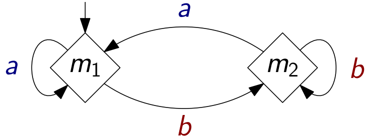
# Finite memory

**Finite-memory strategy $\approx$ memory structure + next-action function.**

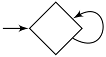
## Memory structure

*Memory structure* $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$: finite set of states $M$, initial state $m_{\mathsf{init}}$, update function $\alpha_{\mathsf{upd}} \colon M \times C \to M$.

Ex.: remember whether $a$ or $b$ was last seen:




Given an arena $\mathcal{A} = (S, S_1, S_2, E)$: *next-action function* $\alpha_{\mathsf{nxt}} \colon S_i \times M \to E$.

**Memoryless** strategies use **memory structure**  $C$ .

# Finite-memory determinacy

## Finite-memory determinacy

An objective $W$ is **finite-memory-determined** if **there exists** **a finite memory structure** $\mathcal{M}$ that suffices to play optimally for both players **in all arenas** $\mathcal{A}$.

## Remark

Usually, the definition inverts the order of the quantifiers. The order has an impact in **finite arenas**,[13] but not in **infinite arenas**.

[13] Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2020.

# Get rid of prefix-independence? Right congruence

Let $L$ be a language of **finite** words on alphabet $C$.

### Myhill-Nerode congruence

For $x, y \in C^*$, $x \sim_L y$ if for all $z \in C^*$, $xz \in L \Leftrightarrow yz \in L$.

### Myhill-Nerode theorem[14]

$L$ is **regular** if and only if $\sim_L$ has **finitely many equivalence classes**.
The **equivalence classes of** $\sim_L$ correspond to the **states of the minimal DFA for** $L$.

[14] Nerode, "Linear Automaton Transformations", 1958.

# Get rid of prefix-independence? Right congruence

Let $W$ be a language of **infinite** words (= an objective) on alphabet $C$.

### Right congruence

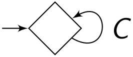
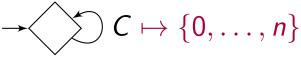
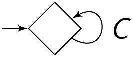
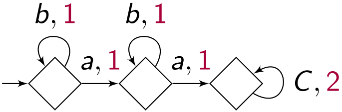
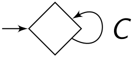
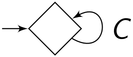
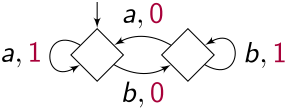
For $x, y \in C^*$, $x \sim_W y$ if for all $z \in C^\omega$, $xz \in W \Leftrightarrow yz \in W$.

### Links with $\omega$-regularity?

- If $W$ is **$\omega$-regular**, then $\sim_W$ has finitely many equivalence classes. In this case, there is a DFA $\mathcal{M}_\sim$ "*prefix-classifier*" associated with $\sim_W$.
- Reciprocal not true.

$W$ is **prefix-independent** if and only if $\sim_W$ has only one equivalence class.

# Four examples

| Objective | Prefix-classifier $\mathcal{M}_\sim$ | Memory |
|---|---|---|
| $C = \{0, \ldots, n\}$, Parity condition |   |   |
| $C = \mathbb{Q}$, $W = \mathsf{MP}^{\geq 0}$ |   | No finite structure |
| $C = \{a, b\}$, $W = b^* a b^* a C^\omega$ |   |   |
| $C = \{a, b\}$, $W = C^*(ab)^\omega$ |   |   |

# Main result

Let $W \subseteq C^\omega$ be an objective.

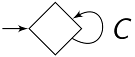
**Theorem**

If a finite memory structure $\mathcal{M}$ suffices to play optimally in infinite arenas for both players, **then**

- ($\mathcal{M}_\sim$ is finite), and
- $W$ **is recognized by a parity automaton** $(\mathcal{M}_\sim \otimes \mathcal{M}, p)$.

$\rightsquigarrow$ if $\mathcal{M}_\sim \otimes \mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$,

$$p \colon M \times C \to \{0, \dots, n\}.$$

Generalizes [CN06][15] (where $\mathcal{M}_\sim = \mathcal{M} =$ 

).

[15] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.

# Corollary

Let $W \subseteq C^\omega$ be an objective.

**Characterization**

$W$ is **finite-memory-determined** if and only if $W$ is $\omega$-**regular**.

**Proof**. $W$ is finite-memory-determined.
$\xRightarrow{[BRV22]}$ $W$ is recognized by a deterministic parity automaton ($\omega$-regular).
$\Longrightarrow$ this parity automaton (as a memory) suffices in infinite arenas.[16]

[16] Zielonka, "Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees", 1998.

# Summary

## Contributions

- **Strategic characterization** of $\omega$-regularity, generalizing [CN06].[17]
- (*Not mentioned*) New **one-to-two-player lift** for zero-sum games on infinite graphs.

## Future work

- Other classes of arenas (e.g., finitely branching)?
- Only **one** player has FM optimal strategies?

# Thanks!

[17] Colcombet and Niwiński, "On the positional determinacy of edge-labeled games", 2006.